

Assignment 2

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PC2135

Thermodynamics and Statistical Mechanics

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Problem 1

[12 pts] Partial derivatives practice

In this class, partial derivatives will be used frequently. In this problem, we do more exercise to familiarize ourselves with partial derivatives.

(1) (3 points) Suppose z is a function of x and y , namely,

$$z(x, y) = xy + x$$

Calculate $\frac{\partial z}{\partial x}_y$ and $\frac{\partial z}{\partial y}_x$.

(2) (4 points) Now define $w = xy$. Then we can view z as a function of x and w . Calculate $\frac{\partial z}{\partial x}_w$ and $\frac{\partial z}{\partial w}_x$.

(3) (5 points) We can also view w as a function of y and z . Calculate $\frac{\partial w}{\partial y}_z$ and $\frac{\partial w}{\partial z}_y$.

Solution

Part (1): Direct partials in (x, y)

From

$$z(x, y) = xy + x,$$

and treating y as a constant (freeze the other coordinate),

$$\frac{\partial z}{\partial x}_y = \frac{\partial xy}{\partial x}_y + \frac{\partial x}{\partial x}_y = y + 1,$$

while treating x as a constant,

$$\frac{\partial z}{\partial y}_x = \frac{\partial xy}{\partial y}_x + \frac{\partial x}{\partial y}_x = x + 0 = x.$$

Part (2): Re-express z as $z(x, w)$

Define $w = xy$, so $y = \frac{w}{x}$. Then

$$z = xy + x = w + x.$$

Holding w fixed (so y adjusts to keep xy constant) gives

$$\frac{\partial z}{\partial x}_w = \left(\frac{\partial(w + x)}{\partial x} \right)_w = 1,$$

and holding x fixed gives

$$\frac{\partial z}{\partial w}_x = \left(\frac{\partial(w + x)}{\partial w} \right)_x = 1.$$

Part (3): Re-express w as $w(y, z)$

From $z = x(y + 1)$,

$$x = \frac{z}{y + 1},$$

so

$$w = xy = \frac{yz}{y + 1}.$$

With z fixed,

$$\frac{\partial w}{\partial y} = \frac{(y + 1)z - yz}{(y + 1)^2} = \frac{z}{(y + 1)^2},$$

and with y fixed,

$$\frac{\partial w}{\partial z} = \frac{y}{y + 1}.$$

Problem 2

[18 pts] Diatomic ideal-gas cycle

Ideal gas is a model that we should understand really really well. In class, we have derived the compression-expansion work and heat of the ideal gas undergoing various processes. In this problem, we strengthen our understanding by applying similar considerations to a diatomic ideal gas undergoing the process shown in Fig. 1. We assume that the energy of this ideal gas is $U = fNk\frac{T}{2}$, where f is the number of degrees of freedom, N is the number of molecules and T is the temperature. In this problem, please express all answers in terms of N , T_h , T_l , f , V_1 and V_2 .

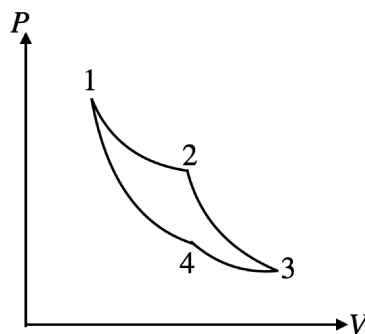


Figure 1: In this process, $1 \rightarrow 2$ is an isothermal process at a temperature T_h , $2 \rightarrow 3$ is an adiabatic process, $3 \rightarrow 4$ is an isothermal process at a temperature T_l , and $4 \rightarrow 1$ is an adiabatic process. Suppose the volumes for the states 1, 2, 3, 4 are V_1, V_2, V_3, V_4 , respectively.

- (1) (2 points) What is V_3 in terms of T_h , T_l and V_2 ? What is V_4 in terms of T_h , T_l and V_1 ?
- (2) (3 points) What is the heat absorbed by the gas during $1 \rightarrow 2$? What is the work done by the gas to the environment during $1 \rightarrow 2$? Note this is not asking about the work done by the environment to the gas.
- (3) (3 points) What is the heat absorbed by the gas during $2 \rightarrow 3$? What is the work done by the gas to the environment during $2 \rightarrow 3$?
- (4) (3 points) What is the heat absorbed by the gas during $3 \rightarrow 4$? What is the work done by the gas to the environment during $3 \rightarrow 4$?

(5) (4 points) In the entire process $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 1$, what is the total work W done by the gas to the environment? In the entire process, the only process in which the gas absorbs heat is the process $1 \rightarrow 2$. Denote this heat by $Q_{1 \rightarrow 2}$. Find the ratio $\frac{W}{Q_{1 \rightarrow 2}}$ in terms of T_h and T_l .

Solution

Part (1): Adiabatic volume relations

For an adiabatic ideal gas,

$$TV^{\frac{2}{f}} = \text{const.}$$

Along $2 \rightarrow 3$ and $4 \rightarrow 1$,

$$T_h V_2^{\frac{2}{f}} = T_l V_3^{\frac{2}{f}}, \quad T_h V_1^{\frac{2}{f}} = T_l V_4^{\frac{2}{f}},$$

so dividing by T_l and raising to $\frac{f}{2}$ gives

$$V_3 = V_2 \left(\frac{T_h}{T_l} \right)^{\frac{f}{2}}, \quad V_4 = V_1 \left(\frac{T_h}{T_l} \right)^{\frac{f}{2}}.$$

Part (2): Isothermal expansion $1 \rightarrow 2$

At constant T_h , the gas expands and does work. Use

$$P = Nk \frac{T_h}{V},$$

$$W_{1 \rightarrow 2} = \int_{V_1}^{V_2} P dV = NkT_h \ln \left(\frac{V_2}{V_1} \right).$$

For an ideal gas $U = \left(\frac{f}{2}\right)NkT$, so $\Delta T = 0$ implies

$$\Delta U = 0,$$

and the first law ($\Delta U = Q - W$) gives

$$Q_{1 \rightarrow 2} = \Delta U + W_{1 \rightarrow 2} = W_{1 \rightarrow 2} = NkT_h \ln \left(\frac{V_2}{V_1} \right).$$

Part (3): Adiabatic expansion $2 \rightarrow 3$

Adiabatic implies no heat exchange:

$$Q_{2 \rightarrow 3} = 0.$$

The temperature drops from T_h to T_l , so

$$\Delta U = \frac{f}{2}Nk(T_l - T_h),$$

and the first law yields

$$W_{2 \rightarrow 3} = -\Delta U = \frac{f}{2}Nk(T_h - T_l).$$

Part (4): Isothermal compression $3 \rightarrow 4$

At constant T_l the environment compresses the gas, so the gas does negative work; use $P = Nk \frac{T_l}{V}$:

$$W_{3 \rightarrow 4} = NkT_l \ln \left(\frac{V_4}{V_3} \right),$$

and $\Delta U = 0$, so by the first law

$$Q_{3 \rightarrow 4} = W_{3 \rightarrow 4} = NkT_l \ln\left(\frac{V_4}{V_3}\right).$$

Part (5): Net work and efficiency ratio

For adiabatic $4 \rightarrow 1$,

$$Q_{4 \rightarrow 1} = 0, \quad W_{4 \rightarrow 1} = -\frac{f}{2}Nk(T_h - T_l).$$

Thus $W_{2 \rightarrow 3} + W_{4 \rightarrow 1} = 0$ (the adiabatic legs cancel). Using

$$\frac{V_4}{V_3} = \frac{V_1}{V_2},$$

the total work is

$$W = W_{1 \rightarrow 2} + W_{3 \rightarrow 4} = Nk(T_h - T_l) \ln\left(\frac{V_2}{V_1}\right).$$

With $Q_{1 \rightarrow 2} = NkT_h \ln\left(\frac{V_2}{V_1}\right)$,

$$\frac{W}{Q_{1 \rightarrow 2}} = 1 - \frac{T_l}{T_h}.$$

Problem 3

[5 pts] Specific heat and latent heat

At 1 atm, the specific heat capacity of liquid water is $4.2 \frac{\text{J}}{\text{gK}}$. The latent heat for melting ice is 333 J g^{-1} , and the latent heat for boiling water is 2260 J g^{-1} . Starting from 1 g of ice at 0°C , how much heat is needed to fully convert the ice to vapor? Assume in this process the ice is first fully melted into water, and then the water increases its temperature until it is boiling, which finally becomes vapor.

Solution

The process is: melt ice at 0°C (latent heat), heat liquid water to 100°C (sensible heat), then vaporize at 100°C (latent heat). The total heat is the sum of these three stages:

$$Q = mL_f + mc\Delta T + mL_v.$$

For $m = 1 \text{ g}$ and $\Delta T = 100 \text{ K}$,

$$Q = 333 \text{ J} + 420 \text{ J} + 2260 \text{ J} = 3013 \text{ J}.$$