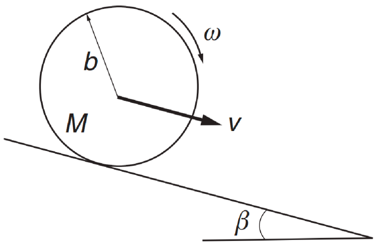
Worksheet 9

Parth Bhargava · A0310667E

PC3261
Classical Mechanics II

November 14, 2025

Problem 4.5



A uniform drum of radius b and mass M rolls without slipping down a plane inclined at angle β. Find the drum's acceleration along the plane.

Translation of the center of mass:

$$\begin{cases} Mg\sin\beta - f = M\ddot{X}_{\text{CM}}(t) \\ N - Mg\cos\beta = M\ddot{Y}_{\text{CM}}(t) \end{cases}$$

Motion with no slipping: the contact is very rough $f \le \mu_s N$

$$\dot{X}_{\text{CM}}(t) = b\dot{\varphi}(t) = b\omega(t) \implies \ddot{X}_{\text{CM}}(t) = b\ddot{\varphi}(t) = b\dot{\omega}(t)$$

Solution

Position and force vectors:

$$\boldsymbol{R}_{\text{CM}}(t) = X_{\text{CM}}(t)\hat{e}_x + b\hat{e}_y$$

$$\begin{cases} \boldsymbol{W}(t) = Mg\sin\beta\hat{e}_x - Mg\cos\beta\hat{e}_y \\ \boldsymbol{f}(t) = -f(t)\hat{e}_x \\ \boldsymbol{N}(t) = N(t)\hat{e}_y \end{cases} \qquad \begin{cases} \boldsymbol{r}_W(t) = X_{\text{CM}}(t)\hat{e}_x + b\hat{e}_y \\ \boldsymbol{r}_f(t) = X_{\text{CM}}(t)\hat{e}_x \\ \boldsymbol{r}_N(t) = X_{\text{CM}}(t)\hat{e}_x \end{cases}$$

From Newton's second law:

$$\boldsymbol{F}(t) = M\ddot{\boldsymbol{R}}_{\text{CM}}(t) \implies \begin{cases} Mg\sin\beta - f(t) = M\ddot{X}_{\text{CM}}(t) \\ N(t) - Mg\cos\beta = 0 \end{cases}$$

Approach 1

Torque about center of mass:

$$\boldsymbol{\tau}_{\text{CM}}(t) = \sum_i [\boldsymbol{r}_i(t) - \boldsymbol{R}_{\text{CM}}(t)] \times \boldsymbol{F}_i(t) = -bf(t)\hat{e}_z$$

Angular momentum about center of mass:

$$\boldsymbol{L}_{\text{CM}}(t) = \boldsymbol{L}^{\text{spin}}(t) = -\frac{1}{2}Mb^2\omega(t)\hat{e}_z$$

Setting $\boldsymbol{\tau}_{\text{CM}}(t) = \dot{\boldsymbol{L}}_{\text{CM}}(t)$:

$$bf(t) = \frac{1}{2}Mb^2\dot{\omega}(t)$$

System of equations:

$$\begin{cases} Mg\sin\beta - f(t) = M\ddot{X}_{\text{CM}}(t) \\ N(t) - Mg\cos\beta = 0 \\ bf(t) = \frac{1}{2}Mb^2\dot{\omega}(t) \\ \ddot{X}_{\text{CM}}(t) = b\dot{\omega}(t) \end{cases} \implies \begin{cases} \ddot{X}_{\text{CM}}(t) = \frac{2}{3}g\sin\beta \\ f(t) = \frac{1}{3}Mg\sin\beta \end{cases}$$

$$\boxed{\ddot{X}_{\text{CM}}(t) = \frac{2}{3}g\sin\beta}$$

Approach 2

Torque about contact point:

$$\boldsymbol{\tau}_{\text{contact}}(t) = \sum_i [\boldsymbol{r}_i(t) - \boldsymbol{r}_{\text{contact}}(t)] \times \boldsymbol{F}_i(t) = -Mgb\sin\beta\hat{e}_z$$

$$\boldsymbol{L}_{\text{contact}}(t) = \boldsymbol{L}^{\text{orbital}}(t) + \boldsymbol{L}^{\text{spin}}(t) = -\left[Mb\dot{X}_{\text{CM}}(t) + \frac{1}{2}Mb^2\omega(t)\right]\hat{e}_z$$

$$\boldsymbol{\tau}_{\text{contact}}(t) = \dot{\boldsymbol{L}}_{\text{contact}}(t)$$

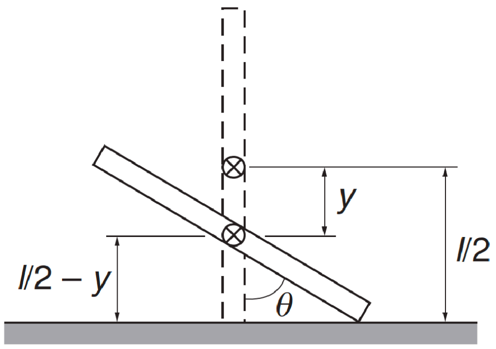
$$\implies Mgb\sin\beta = Mb\ddot{X}_{\text{CM}}(t) + \frac{1}{2}Mb^2\dot{\omega}(t)$$

Final system:

$$\begin{cases} Mg\sin\beta - f(t) = M\ddot{X}_{\text{CM}}(t) \\ N(t) - Mg\cos\beta = 0 \\ Mgb\sin\beta = Mb\ddot{X}_{\text{CM}}(t) + \frac{1}{2}Mb^2\dot{\omega}(t) \\ \ddot{X}_{\text{CM}}(t) = b\dot{\omega}(t) \end{cases} \implies \begin{cases} \ddot{X}_{\text{CM}}(t) = \frac{2}{3}g\sin\beta \\ f(t) = \frac{1}{3}Mg\sin\beta \end{cases}$$

$$\boxed{\ddot{X}_{\text{CM}}(t) = \frac{2}{3}g\sin\beta}$$

Problem 4.6



A uniform stick of length ℓ and mass M, initially upright on a frictionless table, starts falling.

Find the normal force from table as a function of θ from the vertical.

Solution

From Newton's second law:

$$\boldsymbol{F}(t) = M\ddot{\boldsymbol{R}}_{\text{CM}}(t) \implies \begin{cases} 0 = M\ddot{X}_{\text{CM}}(t) \\ N(t) - Mg = M\ddot{Y}_{\text{CM}}(t) \end{cases}$$

Since there are no horizontal forces:

$$\ddot{X}_{\text{CM}}(t) = 0 \implies \dot{X}_{\text{CM}}(t) = \dot{X}_{\text{CM}}(0) = 0 \implies X_{\text{CM}}(t) = X_{\text{CM}}(0) = 0$$

Center of mass position:

$$Y_{\text{CM}}(t) = \frac{\ell}{2}\cos\theta(t) \implies \dot{Y}_{\text{CM}}(t) = -\frac{\ell}{2}\dot{\theta}(t)\sin\theta(t)$$

$$\implies \ddot{Y}_{\text{CM}}(t) = -\frac{\ell}{2}\ddot{\theta}(t)\sin\theta(t) - \frac{\ell}{2}\dot{\theta}^2(t)\cos\theta(t)$$

From the vertical force equation:

$$N(t) = Mg + M\ddot{Y}_{\text{CM}}(t) = Mg - \frac{1}{2}M\ell\left[\ddot{\theta}(t)\sin\theta(t) + \dot{\theta}^2(t)\cos\theta(t)\right]$$

Using torque about center of mass:

$$\tau_{\text{CM}}(t) = I_{\text{CM}}\ddot{\theta}(t) \implies \frac{\ell}{2}N(t)\sin\theta(t) = \frac{1}{12}M\ell^2\ddot{\theta}(t)$$

$$\implies N(t) = \frac{1}{6}M\ell\frac{\ddot{\theta}(t)}{\sin\theta(t)}$$

Equating the two expressions for $N(t)$:

$$\begin{cases} N(t) = Mg - \frac{1}{2}M\ell\left[\ddot{\theta}(t)\sin\theta(t) + \dot{\theta}^2(t)\cos\theta(t)\right] \\ N(t) = \frac{1}{6}M\ell\frac{\ddot{\theta}(t)}{\sin\theta(t)} \end{cases}$$

$$\implies \ddot{\theta}(t) = \frac{6g}{\ell}\sin\theta(t) - 3\sin\theta(t)\left[\ddot{\theta}(t)\sin\theta(t) + \dot{\theta}^2(t)\cos\theta(t)\right]$$

Rearranging:

$$\frac{d}{dt}\left[\dot{\theta}^2(t)\right] = 2\dot{\theta}(t)\ddot{\theta}(t) = 2\dot{\theta}(t)\left\{\frac{6g}{\ell}\sin\theta(t) - 3\sin\theta(t)\left[\ddot{\theta}(t)\sin\theta(t) + \dot{\theta}^2(t)\cos\theta(t)\right]\right\}$$

$$\Longrightarrow \frac{d}{dt}\left[\dot{\theta}^2(t)\right] = -\frac{d}{dt}\left\{\frac{12g}{\ell}\cos\theta(t) + 3\left[\ddot{\theta}(t)\sin\theta(t)\right]^2\right\}$$

Integrating and using initial conditions:

$$\dot{\theta}^2(t) + \frac{12g}{\ell}\cos\theta(t) + 3\left[\dot{\theta}(t)\sin\theta(t)\right]^2 = C$$

At $t = 0$: $\theta(0) = 0$ and $\dot{\theta}(0) = 0$:

$$\dot{\theta}^2(0) + \frac{12g}{\ell}\cos\theta(0) + 3\left[\dot{\theta}(0)\sin\theta(0)\right]^2 = C \Longrightarrow C = \frac{12g}{\ell}$$

Therefore:

$$\dot{\theta}^2(t) + \frac{12g}{\ell}\cos\theta(t) + 3\left[\dot{\theta}(t)\sin\theta(t)\right]^2 = \frac{12g}{\ell}$$

$$\Longrightarrow \dot{\theta}^2(t) = \frac{12g}{\ell}\frac{1-\cos\theta(t)}{1+3\sin^2\theta(t)}$$

From the angular velocity equation:

$$\dot{\theta}^2(t) = \frac{12g}{\ell}\frac{1-\cos\theta(t)}{1+3\sin^2\theta(t)} \Longrightarrow \ddot{\theta}(t) = \frac{6g}{\ell}\sin\theta(t)\frac{4-6\cos\theta(t)+3\cos^2\theta(t)}{\left[1+3\sin^2\theta(t)\right]^2}$$

Substituting back into the normal force equation:

$$N(t) = Mg - \frac{1}{2}M\ell\left[\ddot{\theta}(t)\sin\theta(t) + \dot{\theta}^2(t)\cos\theta(t)\right] = \frac{4-6\cos\theta(t)+3\cos^2\theta(t)}{[1+3\sin\theta(t)]^2}Mg$$

$$\boxed{N(t) = \frac{4-6\cos\theta(t)+3\cos^2\theta(t)}{\left[1+3\sin^2\theta(t)\right]^2}Mg}$$