Worksheet 7

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PC3261
Classical Mechanics II

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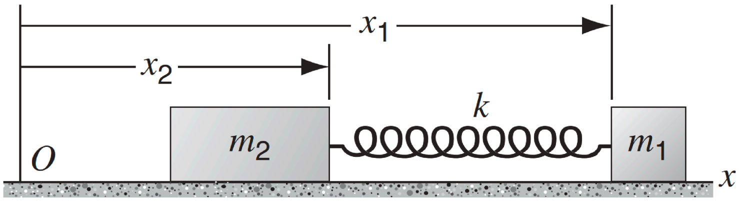
Problem 3.4



Two identical blocks 1 and 2 each of mass m slide without friction on a straight track. They are connected by a massless spring with unstretched length L_0 and spring constant k. Initially, the system is at rest. At $t = 0$, block 1 is hit sharply giving it an instantaneous velocity v_0 to the right.

Find the velocities of each block at later times with respect to the track.

Solution

Center of mass motion:

Since there are no external forces: $\boldsymbol{F}^{\text{ext}}(t) = M\ddot{\boldsymbol{R}}_{\text{CM}}(t) = 0 \rightarrow \ddot{\boldsymbol{R}}_{\text{CM}}(t) = 0$

Center of mass position:

$$X_{\text{CM}}(t) = \frac{m_1x_1(t) + m_2x_2(t)}{m_1 + m_2} = \frac{1}{2}[x_1(t) + x_2(t)]$$

Relative coordinates transformation:

$$\boldsymbol{r}'(t) = \boldsymbol{r}(t) - \boldsymbol{R}_{\text{CM}}(t) \Rightarrow \begin{cases} x_1'(t) = x_1(t) - X_{\text{CM}}(t) = \dfrac{1}{2}[x_1(t) - x_2(t)] \\[2ex] x_2'(t) = x_2(t) - X_{\text{CM}}(t) = -\dfrac{1}{2}[x_1(t) - x_2(t)] \end{cases}$$

Spring extension relation: $x_1(t) - x_2(t) - L_0 = x_1'(t) - x_2'(t) - L_0$

Equations of motion in CM frame:

$$\begin{cases} \boldsymbol{F}_1(t) = m_1\ddot{\boldsymbol{r}}_1'(t) \\ \boldsymbol{F}_2(t) = m_2\ddot{\boldsymbol{r}}_2'(t) \end{cases}$$

$$\Rightarrow \begin{cases} m\ddot{x}_1'(t) = -k[x_1'(t) - x_2'(t) - L_0] \\ m\ddot{x}_2'(t) = +k[x_1'(t) - x_2'(t) - L_0] \end{cases} \xrightarrow{u(t) = x_1'(t) - x_2'(t) - L_0} m\ddot{u}(t) + 2ku(t) = 0$$

General solution:

$$u(t) = A\cos\omega t + B\sin\omega t \quad \text{where } \omega = \sqrt{\frac{2k}{m}}$$

For given initial conditions:

$$\begin{cases} u(0) = x_1(0) - x_2(0) - L_0 = 0 \\ \dot{u}(0) = \dot{x}_1(0) - \dot{x}_2(0) = v_0 \end{cases} \Rightarrow u(t) = \frac{v_0}{\omega} \sin \omega t$$

Velocities in CM frame:

$$\begin{cases} x_1'(t) = \dfrac{1}{2}[x_1(t) - x_2(t)] \\ x_2'(t) = -\dfrac{1}{2}[x_1(t) - x_2(t)] \end{cases} \Rightarrow x_1'(t) = -x_2'(t)$$

Since $u(t) = x_1'(t) - x_2'(t) - L_0$:

$$\dot{u}(t) = \dot{x}_1'(t) - \dot{x}_2'(t) = v_0 \cos \omega t \to \dot{x}_1'(t) = -\dot{x}_2'(t) = \frac{v_0}{2} \cos \omega t$$

Center of mass velocity:

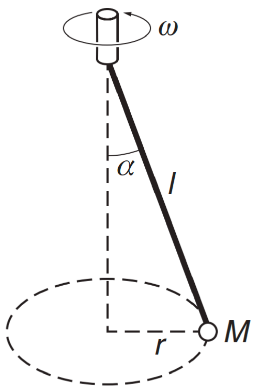
$$\ddot{X}_{\text{CM}}(t) = 0 \to \dot{X}_{\text{CM}}(t) = \dot{X}_{\text{CM}}(0) = \frac{1}{2}[\dot{x}_1(0) + \dot{x}_2(0)] = \frac{v_0}{2}$$

Final velocities:

$$\begin{cases} \dot{x}_1(t) = \dot{x}_1'(t) + \dot{X}_{\text{CM}}(t) = \dfrac{v_0}{2} \cos \omega t + \dfrac{v_0}{2} \\ \dot{x}_2(t) = \dot{x}_2'(t) + \dot{X}_{\text{CM}}(t) = -\dfrac{v_0}{2} \cos \omega t + \dfrac{v_0}{2} \end{cases}$$

$$\boxed{\begin{cases} \dot{x}_1(t) = \dfrac{v_0}{2}(1 + \cos \omega t) \\ \dot{x}_2(t) = \dfrac{v_0}{2}(1 - \cos \omega t) \end{cases}}$$

Problem 4.1



Mass M is fixed to the end of a light rod of length L that is pivoted to swing from the end of a hub that rotates at constant angular frequency ω. The mass moves with steady speed in a circular path of constant radius.

Verify that the relation $\boldsymbol{\tau}(t) = \dot{\boldsymbol{\ell}}(t)$ is satisfied for the following two origins: (1) center of the circular plane of motion; and (2) pivot point on the axis.

Solution

Given: $\boldsymbol{r}(t) = L \sin \alpha \hat{e}_\rho$, $\boldsymbol{F}(t) = -Mg \tan \alpha \hat{e}_\rho$, $\boldsymbol{v}(t) = L\omega \sin \alpha \hat{e}_\varphi$

Case 1: About center of circular motion

Torque about center:

$$\boldsymbol{\tau}(t) = \boldsymbol{r}(t) \times \boldsymbol{F}(t) = \left(L \sin\alpha \hat{e}_\rho\right) \times \left(-Mg \tan\alpha \hat{e}_\rho\right) = 0$$

Angular momentum about center:

$$\ell(t) = \boldsymbol{r}(t) \times \boldsymbol{p}(t) = \left(L \sin\alpha \hat{e}_\rho\right) \times \left(ML\omega \sin\alpha \hat{e}_\varphi\right) = ML^2\omega \sin^2\alpha \hat{e}_z$$

Time derivative:

$$\dot{\ell}(t) = \frac{\mathrm{d}\left(Mr^2\omega\hat{e}_z\right)}{\mathrm{d}t} = 0$$

Verified: $\boldsymbol{\tau}(t) = 0 = \dot{\ell}(t)$

Case 2: About pivot point

Position from pivot: $\boldsymbol{r}_{\text{pivot}}(t) = L \cos\alpha \hat{e}_z$

Torque about pivot:

$$\begin{aligned}
\boldsymbol{\tau}_{\text{pivot}}(t) &= \left[\boldsymbol{r}(t) - \boldsymbol{r}_{\text{pivot}}(t)\right] \times \boldsymbol{F}(t) \\
&= \left[L \sin\alpha \hat{e}_\rho - L \cos\alpha \hat{e}_z\right] \times \left(-Mg \tan\alpha \hat{e}_\rho\right) \\
&= MgL \sin\alpha \hat{e}_\varphi
\end{aligned}$$

Angular momentum about pivot:

$$\begin{aligned}
\ell_{\text{pivot}}(t) &= \left[\boldsymbol{r}(t) - \boldsymbol{r}_{\text{pivot}}(t)\right] \times \boldsymbol{p}(t) \\
&= \left[L \sin\alpha \hat{e}_\rho - L \cos\alpha \hat{e}_z\right] \times \left(ML\omega \sin\alpha \hat{e}_\varphi\right) \\
&= ML^2\omega \sin^2\alpha \hat{e}_z + ML^2\omega \sin\alpha \cos\alpha \hat{e}_\rho
\end{aligned}$$

Time derivative:

$$\dot{\ell}_{\text{pivot}}(t) = ML^2\omega^2 \sin\alpha \cos\alpha \hat{e}_\varphi = MgL \sin\alpha \hat{e}_\varphi$$

Verified: $\boldsymbol{\tau}_{\text{pivot}}(t) = MgL \sin\alpha \hat{e}_\varphi = \dot{\ell}_{\text{pivot}}(t)$