Worksheet 6

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Classical Mechanics II

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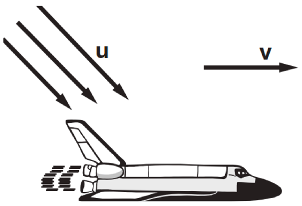
Problem 3.1



A spacecraft with mass M moves through space with constant velocity $\boldsymbol{v}$. The spacecraft encounters a stream of dust particles that embed themselves in the hull at rate $\dot{m}(t)$. The dust has velocity $\boldsymbol{u}$ just before it hits.

Find the external force necessary to keep the spacecraft moving uniformly.

Solution

Using the momentum conservation approach for variable mass systems:

$$\begin{cases} \boldsymbol{P}(t) = M(t)\boldsymbol{v} + (\Delta m)\boldsymbol{u} \\ \boldsymbol{P}(t+\Delta t) = (M(t) + \Delta m)\boldsymbol{v} \end{cases} \Rightarrow \Delta \boldsymbol{P} = \boldsymbol{P}(t+\Delta t) - \boldsymbol{P}(t) = (\boldsymbol{v} - \boldsymbol{u})\Delta m$$

The external force required is:

$$\boxed{\boldsymbol{F}(t) = \frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{P}}{\Delta t} = (\boldsymbol{v} - \boldsymbol{u})\dot{m}(t)}$$

Problem 3.2

A rocket is taking off from rest in a uniform gravitation field $\boldsymbol{g} = -g\hat{e}_z$. The fuel is ejected at a constant rate $\dot{m}(t) = -k$ at a constant exhaust speed u relative to the rocket.

Find $v(t)$ and $r(t)$ for the rocket in its subsequent motion given that the initial mass of the rocket is m_0.

Solution

Given: $\boldsymbol{g} = -g\hat{e}_z$, $\boldsymbol{u} = -u\hat{e}_z$, $\dot{m}(t) = -k \to m(t) = m_0 - kt$

Using the rocket equation:

$$m(t)\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} - \boldsymbol{u}(t)\dot{m}(t) = \boldsymbol{F}^{\text{ext}}(t)$$

$$m(t)\frac{\mathrm{d}v_z}{\mathrm{d}t} + \dot{m}(t)u = -m(t)g$$

$$\frac{\mathrm{d}v_z}{\mathrm{d}t} = -g - u\frac{\dot{m}(t)}{m(t)}$$

$$v_z(t) = -gt - u\ln\frac{m(t)}{m_0}$$

$$v_z(t) = -gt - u\ln\left(\frac{m_0 - kt}{m_0}\right)$$

For position:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = v_z(t)$$

$$z(t) = -\frac{1}{2}gt^2 - u\int_0^t \ln\left(\frac{m_0 - kt'}{m_0}\right)\mathrm{d}t'$$

$$z(t) = -\frac{1}{2}gt^2 + ut + \frac{u}{k}(m_0 - kt)\ln\left(\frac{m_0 - kt}{m_0}\right)$$

$$\boxed{v_z(t) = -gt - u\ln\left(\frac{m_0 - kt}{m_0}\right)}$$

$$\boxed{z(t) = -\frac{1}{2}gt^2 + ut + \frac{u}{k}(m_0 - kt)\ln\left(\frac{m_0 - kt}{m_0}\right)}$$

Problem 3.3

A thin non-uniform plate lies on the xy-plane with corners $(0,0)$, $(a,0)$, $(0,b)$ and (a,b). Its surface mass density is $\sigma(x,y) = \sigma_0\frac{xy}{ab}$ where σ_0 is a constant. Find its center of mass.

Solution

Given: $\boldsymbol{r} = x\hat{e}_x + y\hat{e}_y$, $\sigma(\boldsymbol{r}) = \sigma(x,y) = \sigma_0\frac{xy}{ab}$

First, find the total mass:

$$M = \int\int \sigma(\boldsymbol{r})\,\mathrm{d}A = \int_{x=0}^{a}\int_{y=0}^{b}\frac{\sigma_0}{ab}xy\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{4}\sigma_0 ab$$

Center of mass coordinates:

$$X_{\text{CM}} = \frac{1}{M}\int\int x\sigma(\boldsymbol{r})\,\mathrm{d}A \to X_{\text{CM}} = \frac{1}{M}\int_{x=0}^{a}\int_{y=0}^{b} x\frac{\sigma_0 xy}{ab}\,\mathrm{d}x\,\mathrm{d}y = \frac{2}{3}a$$

$$Y_{\text{CM}} = \frac{1}{M}\int\int y\sigma(\boldsymbol{r})\,\mathrm{d}A \to Y_{\text{CM}} = \frac{1}{M}\int_{x=0}^{a}\int_{y=0}^{b} y\frac{\sigma_0 xy}{ab}\,\mathrm{d}x\,\mathrm{d}y = \frac{2}{3}b$$

$$\boxed{\boldsymbol{R}_{\mathrm{CM}} = \frac{2}{3}a\hat{e}_x + \frac{2}{3}b\hat{e}_y}$$