Worksheet 5

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Classical Mechanics II

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Problem 2.5

A projectile of mass m is launched from the origin with initial velocity v_0 at an angle θ_0 to the horizontal. The projectile experiences a linear drag force with coefficient k and is subject to uniform gravity g.

The equations of motion are:

$$m\ddot{x} = -k\dot{x}, \quad m\ddot{y} = -mg - k\dot{y}$$

with initial conditions $x(0) = y(0) = 0$, $\dot{x}(0) = v_0 \cos\theta_0$, and $\dot{y}(0) = v_0 \sin\theta_0$.

The solutions to these equations are:

$$\dot{x}(t) = v_0 \cos\theta_0 e^{-k\frac{t}{m}}, \quad x(t) = \frac{mv_0 \cos\theta_0}{k}\left(1 - e^{-k\frac{t}{m}}\right)$$

$$\dot{y}(t) = -\frac{mg}{k} + \left(v_0 \sin\theta_0 + \frac{mg}{k}\right)e^{-k\frac{t}{m}}$$

$$y(t) = -\frac{mgt}{k} + \left(\frac{m}{k}\right)\left(v_0 \sin\theta_0 + \frac{mg}{k}\right)\left(1 - e^{-k\frac{t}{m}}\right)$$

The flight time T is determined by the condition $y(T) = 0$, which gives:

$$(\varepsilon \sin\theta_0 + 1)\left(1 - e^{-k\frac{T}{m}}\right) - \frac{kT}{m} = 0$$

where $\varepsilon \equiv \frac{kv_0}{mg}$ is a dimensionless parameter.

Determine the range $R = x(T)$ as a perturbative expansion in ε up to order $\mathcal{O}(\varepsilon^2)$, expressing your answer in the form:

$$R = R_0[1 + a_1\varepsilon + a_2\varepsilon^2 + \mathcal{O}(\varepsilon^3)]$$

where R_0 is the range in the absence of drag.

Solution

Time of flight equation:

$$(\varepsilon \sin\theta_0 + 1)\left(1 - e^{-kT}\right) - kT = 0, \quad \varepsilon \equiv \frac{kv_0}{g}, \quad T = \frac{2v_0 \sin\theta_0}{g}(1 + c_1\varepsilon + c_2\varepsilon^2)$$

$$(\varepsilon \sin\theta_0 + 1)\left(1 - e^{-kT}\right) - kT = 0$$

$$\implies (\varepsilon \sin\theta_0 + 1)\left(1 - e^{-\frac{\varepsilon g T}{v_0}}\right) - \varepsilon\frac{gT}{v_0} = 0$$

$$\implies (\varepsilon \sin\theta_0 + 1)\left\{1 - \mathsf{exp}\left[-\frac{\varepsilon g 2v_0 \sin\theta_0}{v_0 g}(1 + c_1\varepsilon + c_2\varepsilon^2)\right]\right\} - \frac{\varepsilon g 2v_0 \sin\theta_0}{v_0 g}(1 + c_1\varepsilon + c_2\varepsilon^2) = 0$$

$$\implies (\varepsilon \sin\theta_0 + 1)\{1 - \mathsf{exp}[-2\varepsilon \sin\theta_0(1 + c_1\varepsilon + c_2\varepsilon^2)]\} - 2\varepsilon \sin\theta_0(1 + c_1\varepsilon + c_2\varepsilon^2) = 0$$

Define:

$$f(\varepsilon) \equiv (\varepsilon \sin\theta_0 + 1)\{1 - \mathsf{exp}[-2\varepsilon \sin\theta_0(1 + c_1\varepsilon + c_2\varepsilon^2)]\} - 2\varepsilon \sin\theta_0(1 + c_1\varepsilon + c_2\varepsilon^2)$$

Taylor expansion:

$$\begin{cases} f^{(0)}(0) = 0 \\ f^{(1)}(0) = 0 \\ f^{(2)}(0) = 0 \\ f^{(3)}(0) = -4\sin^2\theta_0(3c_1 + \sin\theta_0) \\ f^{(4)}(0) = 16\sin^2\theta_0(\sin^2\theta_0 - 3c_1^2 - 3c_2) \end{cases}$$

$$\Longrightarrow f(\varepsilon) = \frac{1}{6}(-4\sin^2\theta_0)(3c_1 + \sin\theta_0)\varepsilon^3 + \frac{1}{24}(16\sin^2\theta_0)(\sin^2\theta_0 - 3c_1^2 - 3c_2)\varepsilon^4 + \mathcal{O}(\varepsilon^5)$$

Setting $f(\varepsilon) = 0$:

$$\begin{cases} 3c_1 + \sin\theta_0 = 0 \\ \sin^2\theta_0 - 3c_1^2 - 3c_2 = 0 \end{cases} \Longrightarrow \begin{cases} c_1 = -\dfrac{1}{3}\sin\theta_0 \\ c_2 = \dfrac{2}{9}\sin^2\theta_0 \end{cases}$$

Range calculation:

$$R = \frac{v_0\cos\theta_0}{k}(1 - e^{-kT}), \quad \varepsilon \equiv \frac{kv_0}{g}, \quad T = \frac{2v_0\sin\theta_0}{g}(1 + c_1\varepsilon + c_2\varepsilon^2)$$

$$R = \frac{v_0\cos\theta_0}{k}(1 - e^{-kT}) = \frac{v_0^2\cos\theta_0}{\varepsilon g}\left(1 - e^{-\frac{\varepsilon gT}{v_0}}\right)$$

$$= \frac{v_0^2\cos\theta_0}{\varepsilon g}\left\{1 - \mathsf{exp}\left[-\frac{\varepsilon g 2v_0\sin\theta_0}{v_0 g}(1 + c_1\varepsilon + c_2\varepsilon^2)\right]\right\}$$

$$= \frac{v_0^2\cos\theta_0}{\varepsilon g}\{1 - \mathsf{exp}[-2\varepsilon\sin\theta_0(1 + c_1\varepsilon + c_2\varepsilon^2)]\}$$

Define:

$$g(\varepsilon) \equiv \frac{v_0^2\cos\theta_0}{\varepsilon g}\{1 - \mathsf{exp}[-2\varepsilon\sin\theta_0(1 + c_1\varepsilon + c_2\varepsilon^2)]\}$$

Taylor expansion of $g(\varepsilon)$:

$$\begin{cases} g^{(0)}(0) = \dfrac{2v_0^2\sin\theta_0\cos\theta_0}{g} \\ g^{(1)}(0) = -\dfrac{8v_0^2\sin^2\theta_0\cos\theta_0}{3g} \\ g^{(2)}(0) = \dfrac{56v_0^2\sin^3\theta_0\cos\theta_0}{9g} \end{cases}$$

Using $c_1 = -\frac{1}{3}\sin\theta_0$ and $c_2 = \frac{2}{9}\sin^2\theta_0$:

$$\boxed{R = \frac{2v_0^2\sin\theta_0\cos\theta_0}{g}\left[1 - \frac{4}{3}\varepsilon\sin\theta_0 + \frac{14}{9}\varepsilon^2\sin^2\theta_0 + \mathcal{O}(\varepsilon^3)\right]}$$

Problem 2.6

A charged particle of mass m and charge q moves in a uniform magnetic field $\boldsymbol{B} = B_0\hat{e}_y$. The equations of motion in the xz plane are:

$$\begin{cases} \ddot{x}(t) = -\omega\dot{z}(t) \\ \ddot{z}(t) = \omega\dot{x}(t) \end{cases}$$

where $\omega \equiv \frac{qB_0}{m}$ is the cyclotron frequency.

Consider the proposed solutions:

$$\begin{cases} x(t) = C_1\cos\omega t + C_2\sin\omega t + C_0 \\ z(t) = D_1\cos\omega t + D_2\sin\omega t + D_0 \end{cases}$$

(a) Show that the six constants $(C_0, C_1, C_2, D_0, D_1, D_2)$ are not all independent by deriving the coupling constraints imposed by the equations of motion.

(b) Apply the initial conditions $x(0) = x_0$, $\dot{x}(0) = 0$, $z(0) = z_0$, and $\dot{z}(0) = v_{z0}$ to determine the complete solution for $x(t)$ and $z(t)$.

Solution

Decoupling method:

Starting with the coupled system:

$$\begin{cases} \ddot{x}(t) = -\omega\dot{z}(t) \\ \ddot{z}(t) = \omega\dot{x}(t) \end{cases} \implies \begin{cases} \dddot{x}(t) = -\omega^2\dot{x}(t) \\ \dddot{z}(t) = -\omega^2\dot{z}(t) \end{cases}$$

Characteristic equation method:

For $x(t) = e^{\lambda t}$:

$$\dddot{x}(t) = -\omega^2\dot{x}(t) \implies \lambda(\lambda^2 + \omega^2) = 0 \implies \lambda = 0, \pm i\omega$$

General solution using complex exponentials:

$$x(t) = Ae^{+i\omega t} + B + Ce^{-i\omega t} = (A + C)\cos\omega t + i(A - C)\sin\omega t + B$$

Coupling constraints:

From the original equations: $\ddot{x} = -\omega\dot{z}$

Differentiating $x(t) = C_1\cos\omega t + C_2\sin\omega t + C_0$:

$$\dot{x}(t) = -C_1\omega\sin\omega t + C_2\omega\cos\omega t$$

$$\ddot{x}(t) = -C_1\omega^2\cos\omega t - C_2\omega^2\sin\omega t$$

For $z(t) = D_1\cos\omega t + D_2\sin\omega t + D_0$:

$$\dot{z}(t) = -D_1\omega\sin\omega t + D_2\omega\cos\omega t$$

Substituting into $\ddot{x} = -\omega\dot{z}$:

$$-C_1\omega^2\cos\omega t - C_2\omega^2\sin\omega t = \omega^2 D_1\sin\omega t - \omega^2 D_2\cos\omega t$$

Comparing coefficients: $C_1 = D_2$ and $C_2 = -D_1$

$$\begin{cases} x(t) = C_1\cos\omega t + C_2\sin\omega t + C_0 \\ z(t) = -C_2\cos\omega t + C_1\sin\omega t + D_0 \end{cases}$$

where only 4 constants (C_0, C_1, C_2, D_0) are independent due to coupling constraints.

With initial conditions $x(0) = x_0, \dot{x}(0) = 0, z(0) = z_0, \dot{z}(0) = v_{z0}$:

$$\begin{cases} x(0) = C_1 + C_0 = x_0 \\ \dot{x}(0) = C_2\omega = 0 \\ z(0) = -C_2 + D_0 = z_0 \\ \dot{z}(0) = C_1\omega = v_{z0} \end{cases}$$

Solving: $C_2 = 0, C_1 = \frac{v_{z0}}{\omega}, C_0 = x_0 - \frac{v_{z0}}{\omega}, D_0 = z_0$

$$\boxed{\begin{cases} x(t) = x_0 + \dfrac{v_{z0}}{\omega}(\cos\omega t - 1) \\ z(t) = z_0 + \dfrac{v_{z0}}{\omega}\sin\omega t \end{cases}}$$

This describes circular motion in the xz plane with cyclotron frequency ω.