Worksheet 4

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PC3261
Classical Mechanics II

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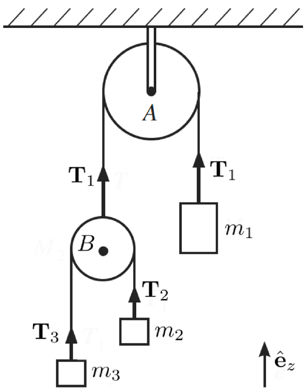
Problem 2.1



A mass m_1 hangs at one end of a string that is led over a pulley A. The other end carries another pulley B which in turn carries a string with the masses m_2 and m_3 fixed to its ends. All pulleys and strings are assumed to be massless. Also, all strings are inextensible.

Find the acceleration of the masses.

Solution

Constraint analysis:

Let a_1, a_2, a_3 be accelerations of masses m_1, m_2, m_3 respectively.

From string constraints:

$$\begin{cases} a_{1A} = -a_{BA} \\ a_{2B} = -a_{3B} \end{cases}$$

$$a_{2B} = -a_{3B} \implies a_2 + a_1 = -(a_3 + a_1) \implies a_1 = -\frac{1}{2}(a_2 + a_3)$$

Also since the strings are assumed to be massless and inextensible:

$$T_1 = 2T_2 = 2T_3 = 2T$$

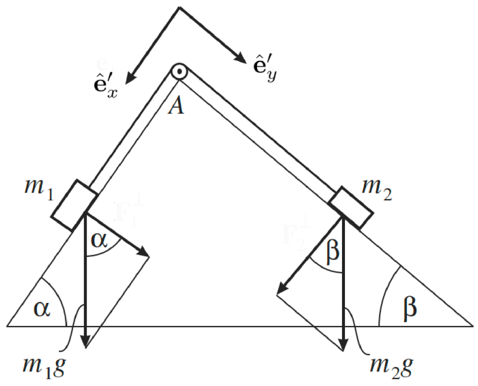
Force equations:

$$\begin{cases} T_1 - m_1 g = m_1 a_1 \\ T_2 - m_2 g = m_2 a_2 \\ T_3 - m_3 g = m_3 a_3 \end{cases} \implies \begin{cases} 2T - m_1 g = m_1 \dfrac{a_2 + a_3}{2} \\ T - m_2 g = m_2 a_2 \\ T - m_3 g = m_3 a_3 \end{cases}$$

Solution:

$$
\begin{cases}
a_2 = -\dfrac{4m_2m_3 + m_1(m_2 - 3m_3)}{m_1(m_2 + m_3) + 4m_2m_3} g \\
a_3 = -\dfrac{4m_2m_3 + m_1(m_3 - 3m_2)}{m_1(m_2 + m_3) + 4m_2m_3} g \\
T = \dfrac{4m_1m_2m_3}{m_1(m_2 + m_3) + 4m_2m_3} g \\
a_1 = -\dfrac{1}{2}(a_2 + a_3) = \dfrac{4m_2m_3 - m_1(m_2 + m_3)}{m_1(m_2 + m_3) + 4m_2m_3} g
\end{cases}
$$

Problem 2.2



Two masses m_1 and m_2 are lying each on one of two joined inclined planes with angles α and β with the horizontal. Both inclined planes and the horizontal make a right-angle triangle. The two masses are connected by a massless and inextensible string running over a massless and fixed pulley. The coefficients of kinetic friction of both planes are μ_k.

Find the acceleration of the masses.

Solution

Force analysis for mass m_1:

$$\boldsymbol{F}_1 = (m_1 g \sin\alpha - T - \mu_k N_1)\hat{e}_x + (m_1 g \cos\alpha - N_1)\hat{e}_y$$

$$\boldsymbol{F}_1 = m_1 \boldsymbol{a}_1 \implies \begin{cases} m_1 g \sin\alpha - T - \mu_k N_1 = m_1 a \\ m_1 g \cos\alpha - N_1 = 0 \end{cases}$$

$$\implies m_1 g \sin\alpha - T - \mu_k m_1 g \cos\alpha = m_1 a$$

Force analysis for mass m_2:

$$\boldsymbol{F}_2 = (m_2 g \cos\beta - N_2)\hat{e}_x + (m_2 g \sin\beta - T + \mu_k N_2)\hat{e}_y$$

$$\boldsymbol{F}_2 = m_2 \boldsymbol{a}_2 \implies \begin{cases} m_2 g \cos\beta - N_2 = 0 \\ m_2 g \sin\beta - T + \mu_k N_2 = -m_2 a \end{cases}$$

$$\implies m_2 g \sin\beta - T + \mu_k m_2 g \cos\beta = -m_2 a$$

System of equations:

$$\begin{cases} m_1 g \sin\alpha - T - \mu_k m_1 g \cos\alpha = m_1 a \\ m_2 g \sin\beta - T + \mu_k m_2 g \cos\beta = -m_2 a \end{cases}$$

Solution:

$$\boxed{\begin{cases} a = \dfrac{(m_1 \sin\alpha - m_2 \sin\beta) - \mu_k (m_1 \cos\alpha + m_2 \cos\beta)}{m_1 + m_2} g \\ T = \dfrac{m_1 m_2 g}{m_1 + m_2}[(\sin\alpha + \sin\beta) - \mu_k(\cos\alpha - \cos\beta)] \end{cases}}$$

Special cases:

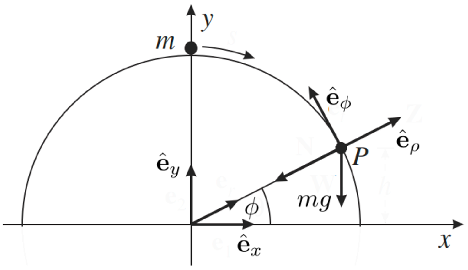
When $\mu_k \to 0$:

$$\boxed{a \to \frac{m_1 \sin\alpha - m_2 \sin\beta}{m_1 + m_2} g}$$

When $\alpha = \beta = \frac{\pi}{2}$:

$$\boxed{a \to \frac{m_1 - m_2}{m_1 + m_2} g}$$

Problem 2.3



A particle of mass m is located at the "North pole" of a smooth hemisphere of radius R fixed on the ground. The particle slides down the hemisphere after a small kick.

Find the angle and the speed at which the particle breaks off from the hemisphere.

Solution

Energy conservation:

$$E = \frac{1}{2}mv^2 + mgR\cos\varphi = mgR$$

$$v^2 = 2gR(1-\cos\varphi)$$

$$\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2 = \frac{2g}{R}(1-\cos\varphi)$$

Radial force equation:

$$N - mg\cos\varphi = mR\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2$$

Breaking condition ($N = 0$):

$$g\cos\varphi = R\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2$$

Substituting the energy result:

$$g\cos\varphi = 2g(1-\cos\varphi)$$

$$3\cos\varphi = 2$$

$$\cos\varphi = \frac{2}{3}$$

Results:

Breaking angle:

$$\boxed{\varphi_0 = \arccos\left(\frac{2}{3}\right) \cong 48.2°}$$

Breaking speed:

$$\boxed{v_0 = \sqrt{\frac{2gR}{3}}}$$

Problem 2.4

Obtain short-time and long-time behaviors for $v_y(t)$ and $y(t)$ for projectile with linear air resistance.

$$\frac{\mathrm{d}v_y}{\mathrm{d}t} = -kv_y(t), \quad y(0) = y_0, \quad v_y(0) = v_0 \cos\theta_0$$

Solution

Differential equation:

$$\frac{\mathrm{d}v_y}{\mathrm{d}t} = -kv_y(t), \quad y(0) = y_0, \quad v_y(0) = v_0 \cos\theta_0$$

Solving for velocity:

$$\frac{\mathrm{d}v_y(t)}{\mathrm{d}t} = -kv_y(t) \implies \int_{v_{y'}=v_0\cos\theta_0}^{v_y} \frac{\mathrm{d}v_{y'}}{v_{y'}} = -k\int_{t'=0}^{t} \mathrm{d}t$$

$$\implies v_y(t) = v_0 \cos\theta_0 e^{-kt}$$

Solving for position:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = v_0\cos\theta_0 e^{-kt} \implies \int_{y'=y_0}^{y} \mathrm{d}y' = \int_{t'=0}^{t} v_0\cos\theta_0 e^{-kt'}\,\mathrm{d}t'$$

$$\implies y(t) = y_0 + \frac{v_0\cos\theta_0}{k}\left(1 - e^{-kt}\right)$$

General solutions:

$$\boxed{v_y(t) = v_0\cos\theta_0 e^{-kt} \quad \text{and} \quad y(t) = y_0 + \frac{v_0\cos\theta_0}{k}\left(1 - e^{-kt}\right)}$$

Short-time behavior ($t \ll \frac{1}{k}$):

$$\begin{cases} v_y(t) \to v_0\cos\theta_0(1 - kt) \\ y(t) \to y_0 + v_0(\cos\theta_0)t - \dfrac{1}{2}kv_0(\cos\theta_0)t^2 \end{cases}$$

Long-time behavior ($t \gg \frac{1}{k}$):

$$\begin{cases} v_y(t) \to 0 \\ y(t) \to y_0 + \dfrac{v_0\cos\theta_0}{k} \end{cases}$$

Alternative form using chain rule:

$$\frac{\mathrm{d}v_y(t)}{\mathrm{d}t} = -kv_y(t) \implies \frac{\mathrm{d}v_y(y)}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}t} = -kv_y(t)$$

$$\implies \int_{v_{y'}=v_0\cos\theta_0}^{v_y} \mathrm{d}v_{y'} = -\int_{y'=y_0}^{y} k\,\mathrm{d}y' \implies v_y(y) = v_0\cos\theta_0 - k(y - y_0)$$