Worksheet 24

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Classical Mechanics II

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Problem 9.9

Obtain the canonical transformation generated by $\Lambda_1(q, Q, t) = qQ$ and the Kamiltonian equations of motion.

Harmonic oscillator Hamiltonian:

$$H(q, p) = \frac{1}{2}m\omega^2 q^2 + \frac{p^2}{2m}$$

Solution

Type 1 generating function:

Given $\Lambda_1(q, Q, t) = qQ$.

For a Type 1 generating function $F_1(q, Q, t)$, the canonical transformation is obtained from:

$$p = \frac{\partial F_1}{\partial q} \quad \text{and} \quad P = -\frac{\partial F_1}{\partial Q}$$

Derive transformation equations:

$$\Lambda_1(q, Q, t) = qQ$$

$$p = \frac{\partial \Lambda_1}{\partial q} = Q$$

$$P = -\frac{\partial \Lambda_1}{\partial Q} = -q$$

To find the inverse transformation $(q, p) \to (Q, P)$, we solve:

$$Q = p$$

and

$$P = -q$$

, which gives:

$$Q(q, p) = p$$

and

$$P(q, p) = -q$$

However, the standard convention is to express (Q, P) in terms of (q, p). Using the phase space Lagrangian gauge transformation:

$$P\dot{Q} - K = p\dot{q} - H + \frac{\mathrm{d}\Lambda_1}{\mathrm{d}t}$$

where:

$$\frac{\mathrm{d}\Lambda_1}{\mathrm{d}t} = \frac{\partial \Lambda_1}{\partial q}\dot{q} + \frac{\partial \Lambda_1}{\partial Q}\dot{Q} + \frac{\partial \Lambda_1}{\partial t} = Q\dot{q} + q\dot{Q}$$

Substituting:

$$P\dot{Q} - K = p\dot{q} - H + Q\dot{q} + q\dot{Q}$$

Rearranging:

$$P\dot{Q} - K = (p + Q)\dot{q} + q\dot{Q} - H$$

For this to hold for all $\dot{q}$ and $\dot{Q}$, we must have:

$$\begin{cases} p + Q = 0 \quad \Rightarrow \quad Q = -p \\ P - q = 0 \quad \Rightarrow \quad P = q \end{cases}$$

Therefore:

$$\boxed{\begin{cases} Q(q,p,t) = -p \\ P(q,p,t) = q \end{cases}}$$

Kamiltonian:

$$K(Q,P,t) = H(q(Q,P), p(Q,P), t) - \frac{\partial \Lambda_1}{\partial t}$$

Since $\frac{\partial \Lambda_1}{\partial t} = 0$, $q = P$, and $p = -Q$:

$$K(Q,P,t) = \frac{1}{2}m\omega^2 P^2 + \frac{Q^2}{2m}$$

$$\boxed{K(Q,P,t) = \frac{1}{2}m\omega^2 P^2 + \frac{Q^2}{2m}}$$

Hamilton's equations in new coordinates:

$$\begin{cases} \dot{Q} = \frac{\partial K}{\partial P} = m\omega^2 P \\ \dot{P} = -\frac{\partial K}{\partial Q} = -\frac{Q}{m} \end{cases}$$

Taking time derivatives:

$$\ddot{Q} = m\omega^2 \dot{P} = m\omega^2 \left(-\frac{Q}{m}\right) = -\omega^2 Q$$

$$\ddot{P} = -\frac{\dot{Q}}{m} = -\frac{m\omega^2 P}{m} = -\omega^2 P$$

$$\boxed{\begin{cases} \dot{Q} = m\omega^2 P \\ \dot{P} = -\frac{Q}{m} \end{cases} \Rightarrow \begin{cases} \ddot{Q} = -\omega^2 Q \\ \ddot{P} = -\omega^2 P \end{cases}}$$

Both Q and P satisfy the harmonic oscillator equation. The transformation exchanges the roles of coordinate and momentum, revealing the symmetry $H(q,p) = K(P,Q)$ (up to constants).

Problem 9.10

Solve for $q(t)$ and $p(t)$ via $Q(t)$ and $P(t)$ using the canonical transformation:

$$\begin{cases} q(Q,P,t) = \sqrt{2\dfrac{P}{m\omega}}\sin Q \\ p(Q,P,t) = \sqrt{2m\omega P}\cos Q \end{cases}$$

with Kamiltonian $K(Q,P,t) = \omega P$.

Solution

Verify canonicality via Poisson brackets:

The transformation is canonical if:

$$\{q,p\}_{Q,P} = 1, \quad \{q,q\}_{Q,P} = 0, \quad \{p,p\}_{Q,P} = 0$$

Compute partial derivatives:

$$\begin{cases} \frac{\partial q}{\partial Q} = \sqrt{2\frac{P}{m\omega}}\cos Q \\ \frac{\partial q}{\partial P} = \frac{1}{\sqrt{2m\omega P}}\sin Q \\ \frac{\partial p}{\partial Q} = -\sqrt{2m\omega P}\sin Q \\ \frac{\partial p}{\partial P} = \sqrt{m\frac{\omega}{2P}}\cos Q \end{cases}$$

Check $\{q,p\}_{Q,P}$:

$$\begin{aligned} \{q,p\}_{Q,P} &= \frac{\partial q}{\partial Q}\frac{\partial p}{\partial P} - \frac{\partial q}{\partial P}\frac{\partial p}{\partial Q} \\ &= \sqrt{2\frac{P}{m\omega}}\cos Q \cdot \sqrt{m\frac{\omega}{2P}}\cos Q - \frac{1}{\sqrt{2m\omega P}}\sin Q \cdot \left(-\sqrt{2m\omega P}\sin Q\right) \\ &= \cos^2 Q + \sin^2 Q = 1 \quad \checkmark \end{aligned}$$

$\{q,q\}_{Q,P} = 0$ and $\{p,p\}_{Q,P} = 0$ trivially. The transformation is canonical.

Solve Kamiltonian equations:

$$K(Q,P,t) = \omega P$$

$$\begin{cases} \dot{Q} = \frac{\partial K}{\partial P} = \omega \\ \dot{P} = -\frac{\partial K}{\partial Q} = 0 \end{cases}$$

Integrating:

$$\boxed{\begin{cases} Q(t) = \omega t + Q(0) \\ P(t) = P(0) \end{cases}}$$

Transform back to (q,p):

$$\begin{cases} q(t) = \sqrt{2\frac{P(0)}{m\omega}}\sin[\omega t + Q(0)] \\ p(t) = \sqrt{2m\omega P(0)\cos[\omega t + Q(0)]} \end{cases}$$

$$\boxed{\begin{cases} q(t) = \sqrt{2\dfrac{P(0)}{m\omega}}\sin[\omega t + Q(0)] \\ p(t) = \sqrt{2m\omega P(0)}\cos[\omega t + Q(0)] \end{cases}}$$

Determine constants from initial conditions:

At $t = 0$:

$$\begin{cases} q(0) = \sqrt{2\dfrac{P(0)}{m\omega}}\sin Q(0) \\ p(0) = \sqrt{2m\omega P(0)}\cos Q(0) \end{cases}$$

Squaring and adding:

$$q(0)^2 + \frac{p(0)^2}{m^2\omega^2} = \frac{2P(0)}{m\omega}[\sin^2 Q(0) + \cos^2 Q(0)] = \frac{2P(0)}{m\omega}$$

$$P(0) = \frac{m\omega}{2}\left[q(0)^2 + \frac{p(0)^2}{m^2\omega^2}\right] = \frac{m\omega q(0)^2}{2} + \frac{p(0)^2}{2m\omega}$$

Dividing the equations:

$$\tan Q(0) = \frac{m\omega q(0)}{p(0)} \quad \Rightarrow \quad Q(0) = \arctan\left(\frac{m\omega q(0)}{p(0)}\right)$$

Compact form:

Define amplitude $A = \sqrt{2\frac{P(0)}{m\omega}}$ and phase $\varphi = Q(0)$:

$$\boxed{\begin{cases} q(t) = A\sin(\omega t + \varphi) \\ p(t) = m\omega A\cos(\omega t + \varphi) \end{cases}}$$

where:

$$\boxed{\begin{cases} A = \sqrt{q(0)^2 + \dfrac{p(0)^2}{m^2\omega^2}} \\ \varphi = \arctan\left(\dfrac{m\omega q(0)}{p(0)}\right) \end{cases}}$$

The canonical transformation converts the problem to one with cyclic coordinate Q, where $P = \frac{E}{\omega}$ is conserved (proportional to energy). The coordinate Q evolves uniformly in time, giving the familiar sinusoidal solution.