Worksheet 23

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Classical Mechanics II

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Problem 9.7

Evaluate $\{\boldsymbol{r}, \boldsymbol{n}\cdot\boldsymbol{L}\}_{\boldsymbol{q},\boldsymbol{p}}$ where $\boldsymbol{r} = x\hat{e}_x + y\hat{e}_y + z\hat{e}_z$ and $\boldsymbol{n} = n_x\hat{e}_x + n_y\hat{e}_y + n_z\hat{e}_z$ is a constant vector.

Angular momentum: $\boldsymbol{L} = \boldsymbol{r}\times\boldsymbol{p}$, with components $L_k = \sum_{i,j}\varepsilon_{ijk}x_i p_j$

Solution

Express the Poisson bracket:

We want to calculate:

$$\{x_i, \boldsymbol{n}\cdot\boldsymbol{L}\}_{\boldsymbol{q},\boldsymbol{p}}$$

for each component x_i of $\boldsymbol{r}$.

Expand $\boldsymbol{n}\cdot\boldsymbol{L}$:

$$\boldsymbol{n}\cdot\boldsymbol{L} = \sum_j n_j L_j = \sum_j n_j \sum_{r,s}\varepsilon_{jrs}x_r p_s$$

Calculate Poisson bracket:

$$\begin{aligned}\{x_i, \boldsymbol{n}\cdot\boldsymbol{L}\}_{\boldsymbol{q},\boldsymbol{p}} &= \left\{x_i, \sum_j\sum_{r,s}\varepsilon_{jrs}n_j x_r p_s\right\}_{\boldsymbol{q},\boldsymbol{p}} \\ &= \sum_j\sum_{r,s}\varepsilon_{jrs}n_j\{x_i, x_r p_s\}_{\boldsymbol{q},\boldsymbol{p}}\end{aligned}$$

Use Leibniz rule:

$$\{x_i, x_r p_s\}_{\boldsymbol{q},\boldsymbol{p}} = x_r\{x_i, p_s\}_{\boldsymbol{q},\boldsymbol{p}} + \{x_i, x_r\}_{\boldsymbol{q},\boldsymbol{p}} p_s$$

Apply fundamental Poisson brackets:

$$\begin{cases}\{x_i, p_s\}_{\boldsymbol{q},\boldsymbol{p}} = \delta_{is} \\ \{x_i, x_r\}_{\boldsymbol{q},\boldsymbol{p}} = 0\end{cases}$$

Therefore:

$$\{x_i, x_r p_s\}_{\boldsymbol{q},\boldsymbol{p}} = x_r\delta_{is}$$

Substitute back:

$$\begin{aligned}\{x_i, \boldsymbol{n}\cdot\boldsymbol{L}\}_{\boldsymbol{q},\boldsymbol{p}} &= \sum_j\sum_{r,s}\varepsilon_{jrs}n_j x_r\delta_{is} \\ &= \sum_j\sum_r\varepsilon_{jri}n_j x_r\end{aligned}$$

Recognize as cross product:

The sum $\sum_{j,r}\varepsilon_{jri}n_j x_r$ is the i-th component of $\boldsymbol{n}\times\boldsymbol{r}$.

To see this, recall that the i-th component of $\boldsymbol{n} \times \boldsymbol{r}$ is:

$$(\boldsymbol{n} \times \boldsymbol{r})_i = \sum_{j,r} \varepsilon_{ijr} n_j x_r$$

We have $\sum_{j,r} \varepsilon_{jri} n_j x_r$. Using the property $\varepsilon_{jri} = \varepsilon_{ijr}$ (cyclic permutation):

$$\sum_{j,r} \varepsilon_{jri} n_j x_r = \sum_{j,r} \varepsilon_{ijr} n_j x_r = (\boldsymbol{n} \times \boldsymbol{r})_i$$

Final answer:

$$\boxed{\{\boldsymbol{r}, \boldsymbol{n} \cdot \boldsymbol{L}\}_{\boldsymbol{q},\boldsymbol{p}} = \boldsymbol{n} \times \boldsymbol{r}}$$

Physical interpretation:

This result shows that the angular momentum $\boldsymbol{L}$ generates infinitesimal rotations in phase space. Specifically, the Poisson bracket with $\boldsymbol{n} \cdot \boldsymbol{L}$ gives the infinitesimal rotation of the position vector about the axis $\boldsymbol{n}$.

More formally, if we consider the canonical transformation generated by $\boldsymbol{n} \cdot \boldsymbol{L}$:

$$\delta \boldsymbol{r} = \varepsilon \{\boldsymbol{r}, \boldsymbol{n} \cdot \boldsymbol{L}\}_{\boldsymbol{q},\boldsymbol{p}} = \varepsilon (\boldsymbol{n} \times \boldsymbol{r})$$

This is precisely an infinitesimal rotation by angle ε about the axis $\boldsymbol{n}$, confirming that angular momentum is the generator of rotations in classical mechanics.

Problem 9.8

A projectile with mass m is moving on the vertical xy-plane in a uniform gravitational field.

Hamiltonian:

$$H(x, y, p_x, p_y, t) = \frac{p_x^2 + p_y^2}{2m} + mgy$$

Given functions:

$$\begin{cases} F_1 \equiv y - \frac{p_y t}{m} - \frac{1}{2}gt^2 \\ F_2 \equiv x - \frac{p_x t}{m} \end{cases}$$

Show that F_1 and F_2 are constants of motion and find three other constants of motion.

Solution

Hamilton's equations:

From the Hamiltonian, we have:

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m}, & \dot{p}_x = -\frac{\partial H}{\partial x} = 0 \\ \dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{m}, & \dot{p}_y = -\frac{\partial H}{\partial y} = -mg \end{cases}$$

Check if F_1 is a constant of motion:

A function $F(\{q_i, p_i\}, t)$ is a constant of motion if:

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \{F, H\}_{\boldsymbol{q},\boldsymbol{p}} + \frac{\partial F}{\partial t} = 0$$

For $F_1 = y - \frac{p_y t}{m} - \frac{gt^2}{2}$:

Calculate the partial derivative:

$$\frac{\partial F_1}{\partial t} = -\frac{p_y}{m} - gt$$

Calculate the Poisson bracket:

$$\begin{aligned} \{F_1, H\}_{\boldsymbol{q},\boldsymbol{p}} &= \frac{\partial F_1}{\partial x}\frac{\partial H}{\partial p_x} - \frac{\partial F_1}{\partial p_x}\frac{\partial H}{\partial x} + \frac{\partial F_1}{\partial y}\frac{\partial H}{\partial p_y} - \frac{\partial F_1}{\partial p_y}\frac{\partial H}{\partial y} \\ &= 0 \cdot \frac{p_x}{m} - 0 \cdot 0 + 1 \cdot \frac{p_y}{m} - \left(-\frac{t}{m}\right) \cdot (-mg) \\ &= \frac{p_y}{m} - \frac{t \cdot mg}{m} \\ &= \frac{p_y}{m} - gt \end{aligned}$$

Therefore:

$$\frac{\mathrm{d}F_1}{\mathrm{d}t} = \{F_1, H\}_{\boldsymbol{q},\boldsymbol{p}} + \frac{\partial F_1}{\partial t} = \left(\frac{p_y}{m} - gt\right) + \left(-\frac{p_y}{m} - gt\right) = 0$$

$$\boxed{F_1 \text{ is a constant of motion.}}$$

Check if F_2 is a constant of motion:

For $F_2 = x - \frac{p_x t}{m}$:

$$\frac{\partial F_2}{\partial t} = -\frac{p_x}{m}$$

$$\begin{aligned}
\{F_2, H\}_{\boldsymbol{q},\boldsymbol{p}} &= \frac{\partial F_2}{\partial x}\frac{\partial H}{\partial p_x} - \frac{\partial F_2}{\partial p_x}\frac{\partial H}{\partial x} + \frac{\partial F_2}{\partial y}\frac{\partial H}{\partial p_y} - \frac{\partial F_2}{\partial p_y}\frac{\partial H}{\partial y} \\
&= 1 \cdot \frac{p_x}{m} - \left(-\frac{t}{m}\right) \cdot 0 + 0 \cdot \frac{p_y}{m} - 0 \cdot (-mg) \\
&= \frac{p_x}{m}
\end{aligned}$$

Therefore:

$$\frac{\mathrm{d}F_2}{\mathrm{d}t} = \{F_2, H\}_{\boldsymbol{q},\boldsymbol{p}} + \frac{\partial F_2}{\partial t} = \frac{p_x}{m} - \frac{p_x}{m} = 0$$

$$\boxed{F_2 \text{ is a constant of motion.}}$$

Find three other constants of motion:

Constant 1: The Hamiltonian itself

$$F_3 \equiv H = \frac{p_x^2 + p_y^2}{2m} + mgy$$

Check:

$$\frac{\partial H}{\partial t} = 0 \quad \text{(no explicit time dependence)}$$

$$\{H, H\}_{\boldsymbol{q},\boldsymbol{p}} = 0 \quad \text{(Poisson bracket of any function with itself vanishes)}$$

Therefore:

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \{H, H\}_{\boldsymbol{q},\boldsymbol{p}} + \frac{\partial H}{\partial t} = 0 + 0 = 0$$

$$\boxed{F_3 = H \text{ (total energy) is a constant of motion.}}$$

Constant 2: Horizontal momentum

$$F_4 \equiv p_x$$

From Hamilton's equations, we already know:

$$\dot{p}_x = -\frac{\partial H}{\partial x} = 0$$

$$\boxed{F_4 = p_x \text{ (horizontal momentum) is a constant of motion.}}$$

Constant 3: Modified vertical momentum

$$F_5 \equiv p_y + mgt$$

The total time derivative is:

$$\frac{\mathrm{d}F_5}{\mathrm{d}t} = \{F_5, H\}_{\boldsymbol{q},\boldsymbol{p}} + \frac{\partial F_5}{\partial t}$$

Calculate the partial derivative with respect to time:

$$\frac{\partial F_5}{\partial t} = mg$$

Calculate the Poisson bracket:

$$\begin{aligned}
\{F_5, H\}_{\boldsymbol{q},\boldsymbol{p}} &= \frac{\partial F_5}{\partial x}\frac{\partial H}{\partial p_x} - \frac{\partial F_5}{\partial p_x}\frac{\partial H}{\partial x} + \frac{\partial F_5}{\partial y}\frac{\partial H}{\partial p_y} - \frac{\partial F_5}{\partial p_y}\frac{\partial H}{\partial y} \\
&= 0 \cdot \frac{p_x}{m} - 0 \cdot 0 + 0 \cdot \frac{p_y}{m} - 1 \cdot (mg) \\
&= -mg
\end{aligned}$$

Therefore:

$$\frac{\mathrm{d}F_5}{\mathrm{d}t} = \{F_5, H\}_{\boldsymbol{q},\boldsymbol{p}} + \frac{\partial F_5}{\partial t} = -mg + mg = 0$$

$$\boxed{F_5 = p_y + mgt \text{ is a constant of motion.}}$$

Summary of five constants of motion:

$$\boxed{\begin{cases}
F_1 = y - \dfrac{p_y t}{m} - \dfrac{gt^2}{2} & \text{(vertical position -- free fall)} \\
F_2 = x - \dfrac{p_x t}{m} & \text{(horizontal position -- uniform motion)} \\
F_3 = H = \dfrac{p_x^2 + p_y^2}{2m} + mgy & \text{(total energy)} \\
F_4 = p_x & \text{(horizontal momentum)} \\
F_5 = p_y + mgt & \text{(modified vertical momentum)}
\end{cases}}$$

Physical interpretation: These constants encode the initial conditions and conserved quantities of projectile motion. F_1 and F_2 relate to the initial position, F_4 and F_5 to the initial momentum, and F_3 to the total energy.