Worksheet 22

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Classical Mechanics II

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Problem 9.2

Construct the Hamiltonian for a particle of mass m subjected to a conservative central force field with potential energy $U(r)$ using polar coordinates (r, φ) as generalized coordinates.

Solution

Lagrangian in polar coordinates:

In polar coordinates, the position is $\boldsymbol{r} = r\hat{e}_r$ and the velocity is:

$$\dot{\boldsymbol{r}} = \dot{r}\hat{e}_r + r\dot{\varphi}\hat{e}_\varphi$$

The kinetic energy is:

$$T = \frac{m}{2}\,|\dot{\boldsymbol{r}}|^2 = \frac{m}{2}(\dot{r}^2 + r^2\dot{\varphi}^2)$$

The Lagrangian is:

$$L(r, \varphi, \dot{r}, \dot{\varphi}) = \frac{m}{2}(\dot{r}^2 + r^2\dot{\varphi}^2) - U(r)$$

Compute generalized momenta:

$$\begin{cases} p_r &= \frac{\partial L}{\partial \dot{r}} = m\dot{r} \\ p_\varphi &= \frac{\partial L}{\partial \dot{\varphi}} = mr^2\dot{\varphi} \end{cases}$$

Invert to find velocities:

$$\begin{cases} \dot{r} &= \frac{p_r}{m} \\ \dot{\varphi} &= \frac{p_\varphi}{mr^2} \end{cases}$$

Construct the Hamiltonian:

The Hamiltonian is:

$$H\left(r, \varphi, p_r, p_\varphi\right) = \sum_i p_i\dot{q}_i - L$$

$$H = p_r\dot{r} + p_\varphi\dot{\varphi} - L$$

Substituting:

$$\begin{aligned} H &= p_r\left(\frac{p_r}{m}\right) + p_\varphi\left(\frac{p_\varphi}{mr^2}\right) - \left[\frac{m}{2}\left(\left(\frac{p_r}{m}\right)^2 + r^2\left(\frac{p_\varphi}{mr^2}\right)^2\right) - U(r)\right] \\ &= \frac{p_r^2}{m} + \frac{p_\varphi^2}{mr^2} - \frac{p_r^2}{2m} - \frac{p_\varphi^2}{2mr^2} + U(r) \\ &= \frac{p_r^2}{2m} + \frac{p_\varphi^2}{2mr^2} + U(r) \end{aligned}$$

$$\boxed{H\Big(r, \varphi, p_r, p_\varphi\Big) = \frac{1}{2m}\left[p_r^2 + \frac{p_\varphi^2}{r^2}\right] + U(r)}$$

Verify Hamilton's equations:

$$\begin{cases} \dot{r} &= \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \quad \checkmark \\ \dot{\varphi} &= \frac{\partial H}{\partial p_\varphi} = \frac{p_\varphi}{mr^2} \quad \checkmark \\ \dot{p}_r &= -\frac{\partial H}{\partial r} = \dfrac{p_\varphi^2}{mr^3} - \frac{\mathrm{d}U}{\mathrm{d}r} \\ \dot{p}_\varphi &= -\frac{\partial H}{\partial \varphi} = 0 \quad \text{(conserved angular momentum)} \end{cases}$$

Physical interpretation:

- φ is cyclic in the Hamiltonian ($\frac{\partial H}{\partial \varphi} = 0$)
- Therefore, $p_\varphi = mr^2\dot{\varphi}$ (angular momentum) is conserved
- The term $p_\varphi^2/(2mr^2)$ represents the kinetic energy of rotation
- The term $p_r^2/(2m)$ represents the kinetic energy of radial motion

Problem 9.3

Derive Hamilton's equations of motion starting from the Hamiltonian definition.

Solution

Definition of the Hamiltonian:

For a system with generalized coordinates $\{q_i\}$ and generalized momenta $\{p_i\}$, the Hamiltonian is defined as:

$$H(\{q_i, p_i\}, t) \equiv \sum_i \dot{q}_i p_i - L(\{q_i, \dot{q}_i\}, t)$$

where the generalized momenta are:

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}$$

Differential of the Hamiltonian:

Taking the total differential of H as a function of $\{q_i, p_i, t\}$:

$$\mathrm{d}H = \sum_i \left[\frac{\partial H}{\partial q_i} \mathrm{d}q_i + \frac{\partial H}{\partial p_i} \mathrm{d}p_i \right] + \frac{\partial H}{\partial t} \mathrm{d}t$$

Differential from the definition:

From the definition $H = \sum_i \dot{q}_i p_i - L$, we have:

$$\mathrm{d}H = \sum_i (\dot{q}_i \, \mathrm{d}p_i + p_i \, \mathrm{d}\dot{q}_i) - \mathrm{d}L$$

Now, the differential of the Lagrangian is:

$$\mathrm{d}L = \sum_i \left[\frac{\partial L}{\partial q_i} \mathrm{d}q_i + \frac{\partial L}{\partial \dot{q}_i} \mathrm{d}\dot{q}_i \right] + \frac{\partial L}{\partial t} \mathrm{d}t$$

Use the definition of generalized momentum:

Since $p_i = \frac{\partial L}{\partial \dot{q}_i}$:

$$\mathrm{d}L = \sum_i \left[\frac{\partial L}{\partial q_i} \mathrm{d}q_i + p_i \, \mathrm{d}\dot{q}_i \right] + \frac{\partial L}{\partial t} \mathrm{d}t$$

Substitute into the expression for dH:

$$\begin{aligned} \mathrm{d}H &= \sum_i (\dot{q}_i \, \mathrm{d}p_i + p_i \, \mathrm{d}\dot{q}_i) - \sum_i \left[\frac{\partial L}{\partial q_i} \mathrm{d}q_i + p_i \, \mathrm{d}\dot{q}_i \right] - \frac{\partial L}{\partial t} \mathrm{d}t \\ &= \sum_i \dot{q}_i \, \mathrm{d}p_i - \sum_i \frac{\partial L}{\partial q_i} \mathrm{d}q_i - \frac{\partial L}{\partial t} \mathrm{d}t \end{aligned}$$

Apply Euler-Lagrange equation:

From the Euler-Lagrange equation:

$$\frac{\partial L}{\partial q_i} = \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = \frac{\mathrm{d}}{\mathrm{d}t}(p_i) = \dot{p}_i$$

Substituting:

$$\mathrm{d}H = \sum_i \dot{q}_i \, \mathrm{d}p_i - \sum_i \dot{p}_i \, \mathrm{d}q_i - \frac{\partial L}{\partial t} \mathrm{d}t$$

Compare with the general differential:

Comparing with $\mathrm{d}H = \sum_i \left[\frac{\partial H}{\partial q_i}\,\mathrm{d}q_i + \frac{\partial H}{\partial p_i}\,\mathrm{d}p_i\right] + \frac{\partial H}{\partial t}\,\mathrm{d}t$:

Coefficients of $\mathrm{d}p_i$:

$$\boxed{\frac{\partial H}{\partial p_i} = \dot{q}_i}$$

Coefficients of $\mathrm{d}q_i$:

$$\boxed{\frac{\partial H}{\partial q_i} = -\dot{p}_i \quad \Rightarrow \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}}$$

Coefficients of $\mathrm{d}t$:

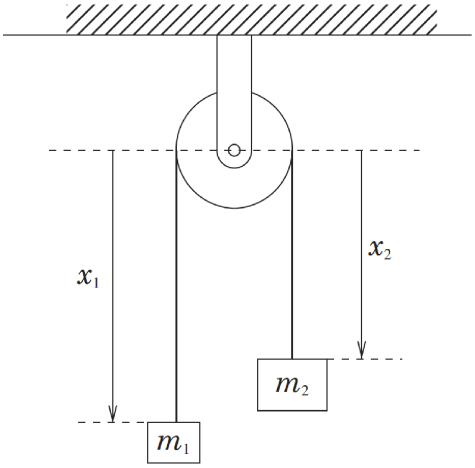
$$\boxed{\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}}$$

Summary: Hamilton's canonical equations of motion

$$\boxed{\begin{cases} \dot{q}_i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} \\ \frac{\partial H}{\partial t} &= -\frac{\partial L}{\partial t} \end{cases}}$$

These $2n$ first-order differential equations are equivalent to the n second-order Euler-Lagrange equations, but they have a beautiful symmetry between coordinates and momenta.

Problem 9.4



Obtain the Hamilton equations of motion for the Atwood machine and solve for the acceleration of the masses.

Recall: Constraint $x_1(t) + x_2(t) = \ell$ already applied, so we use x_1 as the single generalized coordinate.

Solution

Lagrangian (with constraint applied):

From previous analysis:

$$L(x_1, \dot{x}_1) = \frac{1}{2}(m_1 + m_2)\dot{x}_1^2 + (m_1 - m_2)gx_1 + m_2 g\ell$$

(The constant $m_2 g\ell$ can be dropped as it doesn't affect the equations of motion.)

Generalized momentum:

$$p_{x_1} = \frac{\partial L}{\partial \dot{x}_1} = (m_1 + m_2)\dot{x}_1$$

Inverting:

$$\dot{x}_1 = \frac{p_{x_1}}{m_1 + m_2}$$

Construct Hamiltonian:

$$\begin{aligned} H\left(x_1, p_{x_1}\right) &= \dot{x}_1 p_{x_1} - L \\ &= \frac{p_{x_1}}{m_1 + m_2} \cdot p_{x_1} - \left[\frac{1}{2}(m_1 + m_2)\left(\frac{p_{x_1}}{m_1 + m_2}\right)^2 + (m_1 - m_2)gx_1\right] \\ &= \frac{p_{x_1}^2}{m_1 + m_2} - \frac{p_{x_1}^2}{2(m_1 + m_2)} - (m_1 - m_2)gx_1 \end{aligned}$$

$$\boxed{H\left(x_1, p_{x_1}\right) = \frac{p_{x_1}^2}{2(m_1 + m_2)} - (m_1 - m_2)gx_1}$$

Hamilton's equations:

$$\begin{cases} \dot{x}_1 &= \frac{\partial H}{\partial p_{x_1}} = \dfrac{p_{x_1}}{m_1 + m_2} \\ \dot{p}_{x_1} &= -\frac{\partial H}{\partial x_1} = (m_1 - m_2)g \end{cases}$$

Solve for acceleration:

Taking the time derivative of the first equation:

$$\ddot{x}_1 = \frac{\dot{p}_{x_1}}{m_1 + m_2}$$

Substituting the second equation:

$$\ddot{x}_1 = \frac{(m_1 - m_2)g}{m_1 + m_2}$$

$$\boxed{\ddot{x}_1 = \frac{m_1 - m_2}{m_1 + m_2} g}$$

From the constraint $x_2 = \ell - x_1$:

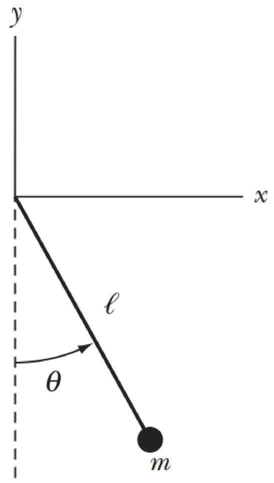
$$\boxed{\ddot{x}_2 = -\ddot{x}_1 = \frac{m_2 - m_1}{m_1 + m_2} g}$$

Physical interpretation:

- If $m_1 > m_2$: $\ddot{x}_1 > 0$ (mass 1 accelerates downward)
- If $m_1 < m_2$: $\ddot{x}_1 < 0$ (mass 1 accelerates upward)
- If $m_1 = m_2$: $\ddot{x}_1 = 0$ (system in equilibrium)

This result is identical to what we obtained from the Lagrangian formulation, confirming the equivalence of the two approaches.

Problem 9.5



Obtain the Hamiltonian equations of motion for the plane pendulum and identify one constant of motion.

Setup: Point mass m on massless rod of length ℓ rotating about frictionless pivot.

Solution

Lagrangian:

Using angle θ from the vertical as the generalized coordinate:

$$L\left(\theta, \dot{\theta}\right) = \frac{m}{2}\ell^2\dot{\theta}^2 + mg\ell\cos\theta$$

Check homogeneity of kinetic energy:

$$T\left(\theta, \lambda\dot{\theta}\right) = \frac{m}{2}\ell^2\left(\lambda\dot{\theta}\right)^2 = \lambda^2 \cdot \frac{m}{2}\ell^2\dot{\theta}^2 = \lambda^2 T\left(\theta, \dot{\theta}\right)$$

The kinetic energy is homogeneous of degree 2 in $\dot{\theta}$. By Euler's theorem:

$$\dot{\theta}\frac{\partial T}{\partial\dot{\theta}} = 2T$$

Generalized momentum:

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = m\ell^2\dot{\theta}$$

Inverting:

$$\dot{\theta} = \frac{p_\theta}{m\ell^2}$$

Construct Hamiltonian:

$$\begin{aligned} H(\theta, p_\theta) &= \dot{\theta}p_\theta - L \\ &= \frac{p_\theta}{m\ell^2} \cdot p_\theta - \left[\frac{m}{2}\ell^2\left(\frac{p_\theta}{m\ell^2}\right)^2 + mg\ell\cos\theta\right] \\ &= \frac{p_\theta^2}{m\ell^2} - \frac{p_\theta^2}{2m\ell^2} - mg\ell\cos\theta \end{aligned}$$

$$\boxed{H(\theta, p_\theta) = \frac{p_\theta^2}{2m\ell^2} - mg\ell\cos\theta}$$

Hamilton's equations:

$$\begin{cases} \dot{\theta} &= \frac{\partial H}{\partial p_\theta} = \dfrac{p_\theta}{m\ell^2} \\ \dot{p}_\theta &= -\frac{\partial H}{\partial \theta} = -mg\ell\sin\theta \end{cases}$$

Recover equation of motion:

Taking the time derivative of the first equation:

$$\ddot{\theta} = \frac{\dot{p}_\theta}{m\ell^2} = \frac{-mg\ell\sin\theta}{m\ell^2} = -\frac{g}{\ell}\sin\theta$$

$$\boxed{\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0}$$

This is the familiar nonlinear pendulum equation.

Identify constant of motion:

Check if H depends explicitly on time:

$$\frac{\partial H}{\partial t} = 0$$

Since H has no explicit time dependence, it is conserved:

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial \theta}\dot{\theta} + \frac{\partial H}{\partial p_\theta}\dot{p}_\theta + \frac{\partial H}{\partial t}$$

Using Hamilton's equations:

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -(\dot{p}_\theta)\dot{\theta} + \left(\dot{\theta}\right)\dot{p}_\theta + 0 = 0$$

$$\boxed{H = \frac{p_\theta^2}{2m\ell^2} - mg\ell\cos\theta = E \quad \text{(total energy is conserved)}}$$

Physical interpretation:

Since the kinetic energy is homogeneous of degree 2:

$$H = T + U = \frac{m}{2}\ell^2\dot{\theta}^2 - mg\ell\cos\theta$$

The Hamiltonian equals the total mechanical energy. The first term is kinetic energy, and the second is potential energy (with the reference at the pivot level).

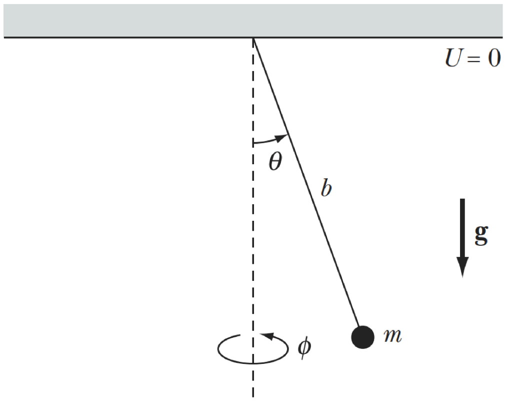
The conservation of energy allows us to reduce the problem to quadrature:

$$\frac{m}{2}\ell^2\dot{\theta}^2 - mg\ell\cos\theta = E$$

$$\dot{\theta} = \pm\sqrt{\frac{2}{m\ell^2}[E + mg\ell\cos\theta]}$$

This can be integrated (using elliptic integrals) to find $\theta(t)$.

Problem 9.6



Obtain equations of motion for the spherical pendulum.

Setup: Bob of mass m moving on sphere of radius b (pendulum length).

Hamiltonian:

$$H\left(\theta, \varphi, p_\theta, p_\varphi\right) = \frac{p_\theta^2}{2mb^2} + \frac{p_\varphi^2}{2mb^2 \sin^2\theta} - mgb\cos\theta$$

Solution

Identify cyclic coordinates:

Note that φ does not appear in the Hamiltonian:

$$\frac{\partial H}{\partial \varphi} = 0$$

Therefore, φ is a cyclic coordinate, and its conjugate momentum is conserved:

$$\dot{p}_\varphi = -\frac{\partial H}{\partial \varphi} = 0 \quad \Rightarrow \quad p_\varphi = \Phi = \text{constant}$$

This is the conservation of angular momentum about the vertical (z) axis.

Reduce to effective one-dimensional problem:

Since $p_\varphi = \Phi$ is constant, we can write the effective Hamiltonian:

$$H_{\text{eff}}(\theta, p_\theta) = \frac{p_\theta^2}{2mb^2} + \frac{\Phi^2}{2mb^2\sin^2\theta} - mgb\cos\theta$$

Hamilton's equations for θ:

$$\begin{cases} \dot{\theta} &= \frac{\partial H}{\partial p_\theta} = \dfrac{p_\theta}{mb^2} \\ \dot{p}_\theta &= -\frac{\partial H}{\partial \theta} = \dfrac{\Phi^2\cos\theta}{mb^2\sin^3\theta} - mgb\sin\theta \end{cases}$$

Equation of motion for θ:

$$\ddot{\theta} = \frac{\dot{p}_\theta}{mb^2} = \frac{\Phi^2\cos\theta}{m^2b^4\sin^3\theta} - \frac{g}{b}\sin\theta$$

$$\boxed{\ddot{\theta} = \frac{\Phi^2\cos\theta}{m^2b^4\sin^3\theta} - \frac{g}{b}\sin\theta}$$

Hamilton's equation for φ:

$$\dot{\varphi} = \frac{\partial H}{\partial p_\varphi} = \frac{p_\varphi}{mb^2\sin^2\theta} = \frac{\Phi}{mb^2\sin^2\theta}$$

$$\boxed{\dot{\varphi} = \frac{\Phi}{mb^2 \sin^2 \theta}}$$

Alternative form using Lagrangian:

From the Lagrangian formulation, we have:

$$L\left(\theta, \varphi, \dot{\theta}, \dot{\varphi}\right) = \frac{m}{2} b^2 \dot{\theta}^2 + \frac{m}{2} b^2 \sin^2 \theta \dot{\varphi}^2 + mgb \cos \theta$$

The Euler-Lagrange equations are:

$$\begin{cases} mb^2 \ddot{\theta} - mb^2 \sin \theta \cos \theta \dot{\varphi}^2 + mgb \sin \theta = 0 \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(mb^2 \sin^2 \theta \dot{\varphi}\right) = 0 \end{cases}$$

Simplifying the first equation:

$$\boxed{\ddot{\theta} = \sin \theta \cos \theta \dot{\varphi}^2 - \frac{g}{b} \sin \theta}$$

And from the second equation (conservation of p_φ):

$$\boxed{mb^2 \sin^2 \theta \dot{\varphi} = \Phi = \text{constant}}$$

Two constants of motion:

1. Angular momentum about vertical axis:
$$C_1 : \quad p_\varphi = mb^2 \sin^2 \theta \dot{\varphi} = \Phi$$

2. Total energy:
$$C_2 : \quad H = \frac{p_\theta^2}{2mb^2} + \frac{\Phi^2}{2mb^2 \sin^2 \theta} - mgb \cos \theta = E$$

These two constants reduce the problem from 4-dimensional phase space $\left(\theta, \varphi, p_\theta, p_\varphi\right)$ to effectively 2-dimensional, making it integrable.

Physical interpretation:

The motion consists of:

- Oscillation in θ (pendulum swinging)
- Precession in φ (azimuthal rotation)

The effective potential for θ motion is:

$$U_{\text{eff}}(\theta) = \frac{\Phi^2}{2mb^2 \sin^2 \theta} - mgb \cos \theta$$

The first term is the centrifugal barrier (prevents $\theta \to 0$), and the second is gravitational potential.