Worksheet 21

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Classical Mechanics II

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Problem 8.9

A simple pendulum of mass m and length ℓ is subjected to a linear resistance force $\boldsymbol{F} = -\gamma \boldsymbol{v}$ with $\gamma > 0$.

Obtain the equation of motion with suitable generalized coordinates.

Solution

Choose generalized coordinate:

We use the angle θ as the generalized coordinate, measured from the vertical downward direction.

Position and velocity in Cartesian coordinates:

$$\begin{cases} x = \ell \sin\theta & \Rightarrow \quad \dot{x} = \ell\dot{\theta}\cos\theta \\ y = -\ell\cos\theta & \Rightarrow \quad \dot{y} = \ell\dot{\theta}\sin\theta \end{cases}$$

Lagrangian:

The velocity is:

$$\boldsymbol{v} = \dot{x}\hat{e}_x + \dot{y}\hat{e}_y = \ell\dot{\theta}\cos\theta\hat{e}_x + \ell\dot{\theta}\sin\theta\hat{e}_y$$

$$|\boldsymbol{v}|^2 = \ell^2\dot{\theta}^2\left(\cos^2\theta + \sin^2\theta\right) = \ell^2\dot{\theta}^2$$

Kinetic energy:

$$T = \frac{m}{2}|\boldsymbol{v}|^2 = \frac{m}{2}\ell^2\dot{\theta}^2$$

Potential energy:

$$U = mgy = -mg\ell\cos\theta$$

Lagrangian:

$$L\left(\theta, \dot{\theta}\right) = \frac{m}{2}\ell^2\dot{\theta}^2 + mg\ell\cos\theta$$

Resistance force:

The resistance force is:

$$\boldsymbol{F}_{\text{res}} = -\gamma\boldsymbol{v} = -\gamma\ell\dot{\theta}\cos\theta\hat{e}_x - \gamma\ell\dot{\theta}\sin\theta\hat{e}_y$$

Generalized force:

The generalized force corresponding to θ is:

$$Q_\theta = \boldsymbol{F}_{\text{res}} \cdot \frac{\partial \boldsymbol{r}}{\partial \theta}$$

where:

$$\frac{\partial \boldsymbol{r}}{\partial \theta} = \frac{\partial}{\partial \theta}\left(x\hat{e}_x + y\hat{e}_y\right) = \ell\cos\theta\hat{e}_x + \ell\sin\theta\hat{e}_y$$

Therefore:

$$\begin{aligned}
\mathcal{Q}_\theta &= \left(-\gamma\ell\dot{\theta}\cos\theta\hat{e}_x - \gamma\ell\dot{\theta}\sin\theta\hat{e}_y\right)\cdot\left(\ell\cos\theta\hat{e}_x + \ell\sin\theta\hat{e}_y\right) \\
&= -\gamma\ell\dot{\theta}\cos\theta\cdot\ell\cos\theta - \gamma\ell\dot{\theta}\sin\theta\cdot\ell\sin\theta \\
&= -\gamma\ell^2\dot{\theta}(\cos^2\theta + \sin^2\theta) \\
&= -\gamma\ell^2\dot{\theta}
\end{aligned}$$

Euler-Lagrange equation with non-conservative force:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \mathcal{Q}_\theta$$

Computing the partial derivatives:

$$\begin{cases} \frac{\partial L}{\partial \dot{\theta}} = m\ell^2\dot{\theta} \\ \frac{\partial L}{\partial \theta} = -mg\ell\sin\theta \end{cases}$$

The equation becomes:

$$m\ell^2\ddot{\theta} - (-mg\ell\sin\theta) = -\gamma\ell^2\dot{\theta}$$

$$m\ell^2\ddot{\theta} + mg\ell\sin\theta = -\gamma\ell^2\dot{\theta}$$

Dividing by $m\ell^2$:

$$\boxed{\ddot{\theta} + \frac{\gamma}{m}\dot{\theta} + \frac{g}{\ell}\sin\theta = 0}$$

This is the equation of motion for a damped pendulum.

For small angles ($\sin\theta$ `approx` θ):

$$\boxed{\ddot{\theta} + \frac{\gamma}{m}\dot{\theta} + \frac{g}{\ell}\theta \texttt{ approx } 0}$$

This is the equation for a damped harmonic oscillator with:

- Damping coefficient: γ/m
- Natural frequency: $\omega_0 = \sqrt{g/\ell}$

Problem 8.10

Show that the Galilean transformation is a gauge transformation for the Lagrangian of a system of N particles interacting via central potentials. Identify the gauge function.

Lagrangian:

$$L(\{\boldsymbol{r}_\alpha, \dot{\boldsymbol{r}}_\alpha\}) = \sum_{\alpha=1}^{N} \frac{m_\alpha}{2} \dot{\boldsymbol{r}}_\alpha \cdot \dot{\boldsymbol{r}}_\alpha - \frac{1}{2} \sum_{\alpha=1}^{N} \sum_{\beta \neq \alpha} U_{\alpha\beta}(|\boldsymbol{r}_\alpha - \boldsymbol{r}_\beta|)$$

Galilean transformation:

$$\boldsymbol{r}_\alpha(t) \to \boldsymbol{r}'_\alpha(t) = \boldsymbol{r}_\alpha(t) + \boldsymbol{V}t$$

Solution

Transform the velocities:

Under the Galilean transformation:

$$\boldsymbol{r}'_\alpha(t) = \boldsymbol{r}_\alpha(t) + \boldsymbol{V}t$$

Taking the time derivative:

$$\dot{\boldsymbol{r}}'_\alpha(t) = \dot{\boldsymbol{r}}_\alpha(t) + \boldsymbol{V}$$

Transform the kinetic energy:

$$\begin{aligned}
T' &= \sum_{\alpha=1}^{N} \frac{m_\alpha}{2} \dot{\boldsymbol{r}}'_\alpha \cdot \dot{\boldsymbol{r}}'_\alpha \\
&= \sum_{\alpha=1}^{N} \frac{m_\alpha}{2} (\dot{\boldsymbol{r}}_\alpha + \boldsymbol{V}) \cdot (\dot{\boldsymbol{r}}_\alpha + \boldsymbol{V}) \\
&= \sum_{\alpha=1}^{N} \frac{m_\alpha}{2} [\dot{\boldsymbol{r}}_\alpha \cdot \dot{\boldsymbol{r}}_\alpha + 2\dot{\boldsymbol{r}}_\alpha \cdot \boldsymbol{V} + \boldsymbol{V} \cdot \boldsymbol{V}] \\
&= \sum_{\alpha=1}^{N} \frac{m_\alpha}{2} \dot{\boldsymbol{r}}_\alpha \cdot \dot{\boldsymbol{r}}_\alpha + \sum_{\alpha=1}^{N} m_\alpha \dot{\boldsymbol{r}}_\alpha \cdot \boldsymbol{V} + \sum_{\alpha=1}^{N} \frac{m_\alpha}{2} \boldsymbol{V} \cdot \boldsymbol{V}
\end{aligned}$$

Transform the potential energy:

The potential depends only on relative positions:

$$|\boldsymbol{r}'_\alpha - \boldsymbol{r}'_\beta| = |(\boldsymbol{r}_\alpha + \boldsymbol{V}t) - \big(\boldsymbol{r}_\beta + \boldsymbol{V}t\big)| = |\boldsymbol{r}_\alpha - \boldsymbol{r}_\beta|$$

Therefore, the potential energy is unchanged:

$$U'(\{\boldsymbol{r}'_\alpha\}) = U(\{\boldsymbol{r}_\alpha\})$$

Transformed Lagrangian:

$$\begin{aligned}
L'(\{\boldsymbol{r}'_\alpha, \dot{\boldsymbol{r}}'_\alpha\}) &= T' - U' \\
&= \sum_{\alpha=1}^{N} \frac{m_\alpha}{2} \dot{\boldsymbol{r}}_\alpha \cdot \dot{\boldsymbol{r}}_\alpha + \sum_{\alpha=1}^{N} m_\alpha \dot{\boldsymbol{r}}_\alpha \cdot \boldsymbol{V} + \sum_{\alpha=1}^{N} \frac{m_\alpha}{2} \boldsymbol{V} \cdot \boldsymbol{V} - U \\
&= L(\{\boldsymbol{r}_\alpha, \dot{\boldsymbol{r}}_\alpha\}) + \sum_{\alpha=1}^{N} m_\alpha \dot{\boldsymbol{r}}_\alpha \cdot \boldsymbol{V} + \sum_{\alpha=1}^{N} \frac{m_\alpha}{2} \boldsymbol{V} \cdot \boldsymbol{V}
\end{aligned}$$

Express in primed coordinates:

Since $\dot{\boldsymbol{r}}_\alpha = \dot{\boldsymbol{r}}'_\alpha - \boldsymbol{V}$ and $\boldsymbol{r}_\alpha = \boldsymbol{r}'_\alpha - \boldsymbol{V}t$:

$$L'(\{\boldsymbol{r}'_\alpha, \dot{\boldsymbol{r}}'_\alpha\}) = L(\{\boldsymbol{r}'_\alpha, \dot{\boldsymbol{r}}'_\alpha\}) + \sum_{\alpha=1}^{N} m_\alpha \dot{\boldsymbol{r}}'_\alpha \cdot \boldsymbol{V} - \sum_{\alpha=1}^{N} m_\alpha \boldsymbol{V} \cdot \boldsymbol{V} + \sum_{\alpha=1}^{N} \frac{m_\alpha}{2} \boldsymbol{V} \cdot \boldsymbol{V}$$

$$L'(\{\boldsymbol{r}'_\alpha, \dot{\boldsymbol{r}}'_\alpha\}) = L(\{\boldsymbol{r}'_\alpha, \dot{\boldsymbol{r}}'_\alpha\}) + \sum_{\alpha=1}^{N} m_\alpha \dot{\boldsymbol{r}}'_\alpha \cdot \boldsymbol{V} - \sum_{\alpha=1}^{N} \frac{m_\alpha}{2} \boldsymbol{V} \cdot \boldsymbol{V}$$

Identify as gauge transformation:

A gauge transformation has the form:

$$L' = L + \frac{\mathrm{d}\Lambda}{\mathrm{d}t}$$

where Λ is the gauge function. We need:

$$\frac{\mathrm{d}\Lambda}{\mathrm{d}t} = \sum_{\alpha=1}^{N} m_\alpha \dot{\boldsymbol{r}}'_\alpha \cdot \boldsymbol{V} - \sum_{\alpha=1}^{N} \frac{m_\alpha}{2} \boldsymbol{V} \cdot \boldsymbol{V}$$

Rewriting:

$$\frac{\mathrm{d}\Lambda}{\mathrm{d}t} = -\sum_{\alpha=1}^{N} m_\alpha \boldsymbol{V} \cdot \dot{\boldsymbol{r}}'_\alpha + \sum_{\alpha=1}^{N} \frac{m_\alpha}{2} \boldsymbol{V} \cdot \boldsymbol{V}$$

Integrate to find gauge function:

$$\frac{\mathrm{d}\Lambda}{\mathrm{d}t} = -\boldsymbol{V} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_{\alpha=1}^{N} m_\alpha \boldsymbol{r}'_\alpha\right] + \sum_{\alpha=1}^{N} \frac{m_\alpha}{2} \boldsymbol{V} \cdot \boldsymbol{V}$$

Integrating:

$$\boxed{\Lambda = -\boldsymbol{V} \cdot \sum_{\alpha=1}^{N} m_\alpha \boldsymbol{r}'_\alpha + \sum_{\alpha=1}^{N} \frac{m_\alpha}{2} (\boldsymbol{V} \cdot \boldsymbol{V}) t + \text{const}}$$

Alternatively, dropping the constant and primes:

$$\boxed{\Lambda(\{\boldsymbol{r}_\alpha\}, t) = -\sum_{\alpha=1}^{N} m_\alpha \boldsymbol{r}_\alpha \cdot \boldsymbol{V} + \frac{1}{2}\left(\sum_{\alpha=1}^{N} m_\alpha\right) |\boldsymbol{V}|^2 \, t}$$

Conclusion: The Galilean transformation is indeed a gauge transformation, meaning that both Lagrangians give the same equations of motion, as they must for physical consistency (the laws of physics are the same in all inertial frames).

Problem 9.1

Starting from $g = g(u, y)$, perform a Legendre transformation to another function $h = h(x, v)$.

Given: $f = f(x, y)$ with $\mathrm{d}f = u\,\mathrm{d}x + v\,\mathrm{d}y$

First transformation ($f \to g$):

$$u = \frac{\partial f}{\partial x}_y \quad \Rightarrow \quad x = x(u, y)$$

$$g(u, y) \equiv f(x(u, y), y) - x(u, y)u$$

$$\mathrm{d}g = -x\,\mathrm{d}u + v\,\mathrm{d}y$$

Solution

Starting point:

We have $g = g(u, y)$ with differential:

$$\mathrm{d}g = -x\,\mathrm{d}u + v\,\mathrm{d}y$$

From this differential, we identify:

$$\begin{cases} x = -\frac{\partial g}{\partial u}_y \\ v = \frac{\partial g}{\partial y}_u \end{cases}$$

Choose new independent variables:

We want to transform from (u, y) to (x, v). This requires:

- u should become a dependent variable (function of x and v)
- y should become a dependent variable (function of x and v)

This means we need to perform two Legendre transformations: one to change $u \to x$ and another to change $y \to v$.

Legendre transformation to change both variables:

The Legendre transformation from $g(u, y)$ to $h(x, v)$ is:

$$h(x, v) \equiv g(u(x, v), y(x, v)) + u(x, v)x - y(x, v)v$$

Calculate the differential of h:

Taking the total differential:

$$\mathrm{d}h = \mathrm{d}g + (u\,\mathrm{d}x + x\,\mathrm{d}u) - (y\,\mathrm{d}v + v\,\mathrm{d}y)$$

Substituting $\mathrm{d}g = -x\,\mathrm{d}u + v\,\mathrm{d}y$:

$$\mathrm{d}h = (-x\,\mathrm{d}u + v\,\mathrm{d}y) + (u\,\mathrm{d}x + x\,\mathrm{d}u) - (y\,\mathrm{d}v + v\,\mathrm{d}y)$$

$$\mathrm{d}h = u\,\mathrm{d}x - y\,\mathrm{d}v$$

Identify partial derivatives:

From $\mathrm{d}h = u\,\mathrm{d}x - y\,\mathrm{d}v$, we have:

$$\begin{cases} u &= \frac{\partial h}{\partial x}_v \\ -y = \frac{\partial h}{\partial v}_x & \Rightarrow \quad y = -\frac{\partial h}{\partial v}_x \end{cases}$$

Summary of transformations:

Starting from $f(x,y)$ with $\mathrm{d}f = u\,\mathrm{d}x + v\,\mathrm{d}y$:

$$\boxed{\begin{cases} \text{First:} & g(u,y) = f - ux, & \mathrm{d}g = -x\,\mathrm{d}u + v\,\mathrm{d}y \\ \text{Second:} & h(x,v) = g + ux - vy, & \mathrm{d}h = u\,\mathrm{d}x - y\,\mathrm{d}v \end{cases}}$$

Combining the two transformations:

$$h(x,v) = [f - ux] + ux - vy = f - vy$$

$$\boxed{h(x,v) = f(x, y(x,v)) - vy(x,v)}$$

with differential $\mathrm{d}h = u\,\mathrm{d}x - y\,\mathrm{d}v$.

Alternative formulation:

We could also write:

$$h(x,v) = -[vy - f(x,y)]$$

or in terms of g:

$$\boxed{h(x,v) = g(u(x,v), y(x,v)) + u(x,v)x - y(x,v)v}$$

Verification:

$$\begin{aligned} \mathrm{d}h &= \frac{\partial g}{\partial u}_y \,\mathrm{d}u + \frac{\partial g}{\partial y}_u \,\mathrm{d}y + u\,\mathrm{d}x + x\,\mathrm{d}u - y\,\mathrm{d}v - v\,\mathrm{d}y \\ &= -x\,\mathrm{d}u + v\,\mathrm{d}y + u\,\mathrm{d}x + x\,\mathrm{d}u - y\,\mathrm{d}v - v\,\mathrm{d}y \\ &= u\,\mathrm{d}x - y\,\mathrm{d}v \quad \checkmark \end{aligned}$$