Worksheet 20

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Classical Mechanics II

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Problem 8.6

Show that the Jacobi energy function $h(\{q_i, \dot{q}_i\}, t)$ is a constant of motion if the Lagrangian does not depend on time explicitly.

The Jacobi energy function is defined as:

$$h(\{q_i, \dot{q}_i\}, t) \equiv \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L(\{q_k, \dot{q}_k\}, t)$$

Solution

Total time derivative of the Lagrangian:

The total time derivative of $L(\{q_k, \dot{q}_k\}, t)$ is:

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \sum_i \left[\frac{\partial L}{\partial q_i} \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i \right] + \frac{\partial L}{\partial t}$$

Apply the Euler-Lagrange equation:

From the Euler-Lagrange equation:

$$\frac{\partial L}{\partial q_i} = \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}_i} \right)$$

Substituting this into the expression above:

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \sum_i \left[\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i \right] + \frac{\partial L}{\partial t}$$

Simplify using product rule:

The first term can be written using the product rule:

$$\sum_i \left[\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i \right] = \sum_i \frac{\mathrm{d}}{\mathrm{d}t} \left[\dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \right]$$

Therefore:

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \left[\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \right] + \frac{\partial L}{\partial t}$$

Time derivative of the Jacobi energy function:

The Jacobi energy function is:

$$h = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$$

Taking its total time derivative:

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i}\right] - \frac{\mathrm{d}L}{\mathrm{d}t}$$

Substituting the expression for $\mathrm{d}L/\,\mathrm{d}t$:

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i}\right] - \frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i}\right] - \frac{\partial L}{\partial t}$$

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{\partial L}{\partial t}$$

Conclusion:

If $\frac{\partial L}{\partial t} = 0$ (Lagrangian has no explicit time dependence), then $\frac{\mathrm{d}h}{\mathrm{d}t} = 0$, and h is a constant of motion.

Additional result: If the kinetic energy is homogeneous of degree 2 in the generalized velocities, i.e., $T(\{q_k, \lambda\dot{q}_k\}) = \lambda^2 T(\{q_k, \dot{q}_k\})$, then by Euler's theorem:

$$\sum_i \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} = 2T$$

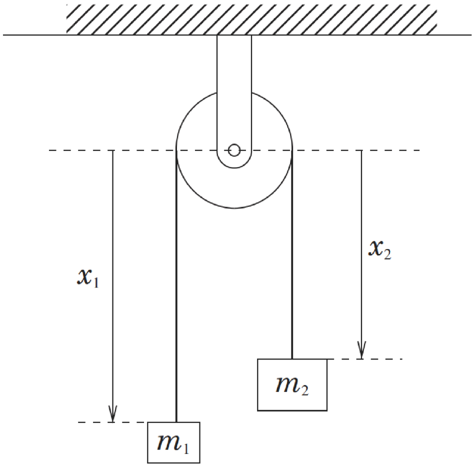
For a system where $L = T - U$ with U independent of velocities:

$$h = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L = \sum_i \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} - (T - U) = 2T - T + U = T + U = E$$

Therefore,

in this case, the Jacobi energy function equals the total mechanical energy.

Problem 8.7



Two masses m_1 and m_2 are suspended by an inextensible string which passes over a massless, frictionless pulley (Atwood machine).

Constraint: $x_1(t) + x_2(t) = \text{constant}$

Kinetic energy: $T = \frac{1}{2}(m_1\dot{x}_1^2 + m_2\dot{x}_2^2)$

Potential energy: $U = -g(m_1x_1 + m_2x_2)$

Solve for the accelerations of the masses from the Euler-Lagrange equation and determine the generalized constraint forces using Lagrange multipliers.

Solution

Method 1: Without Lagrange multipliers (eliminate constraint)

Apply constraint:

From the constraint $x_1(t) + x_2(t) = \ell$ (constant), we have:

$$\dot{x}_2 = -\dot{x}_1, \quad \ddot{x}_2 = -\ddot{x}_1$$

Lagrangian in terms of x_1:

$$T = \frac{1}{2}(m_1\dot{x}_1^2 + m_2\dot{x}_2^2) = \frac{1}{2}(m_1 + m_2)\dot{x}_1^2$$

$$U = -g(m_1x_1 + m_2x_2) = -gm_1x_1 - gm_2(\ell - x_1) = -g(m_1 - m_2)x_1 - gm_2\ell$$

$$L(x_1, \dot{x}_1) = \frac{1}{2}(m_1 + m_2)\dot{x}_1^2 + g(m_1 - m_2)x_1 + gm_2\ell$$

Euler-Lagrange equation:

$$\frac{\partial L}{\partial x_1} = g(m_1 - m_2), \quad \frac{\partial L}{\partial \dot{x}_1} = (m_1 + m_2)\dot{x}_1$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0$$

$$(m_1 + m_2)\ddot{x}_1 - g(m_1 - m_2) = 0$$

$$\boxed{\ddot{x}_1 = \frac{m_1 - m_2}{m_1 + m_2}g, \quad \ddot{x}_2 = -\ddot{x}_1 = \frac{m_2 - m_1}{m_1 + m_2}g}$$

Method 2: With Lagrange multipliers

Set up the system:

The Lagrangian with both coordinates:

$$L(x_1, x_2, \dot{x}_1, \dot{x}_2) = \frac{1}{2}(m_1\dot{x}_1^2 + m_2\dot{x}_2^2) + m_1gx_1 + m_2gx_2$$

Constraint: $\psi(x_1, x_2) = x_1 + x_2 - \ell = 0$

Modified Euler-Lagrange equations:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} &= \lambda \frac{\partial \psi}{\partial x_1} \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} &= \lambda \frac{\partial \psi}{\partial x_2} \\ \psi(x_1, x_2) &= 0 \end{cases}$$

Since $\frac{\partial \psi}{\partial x_1} = 1$ and $\frac{\partial \psi}{\partial x_2} = 1$:

$$\begin{cases} m_1\ddot{x}_1 - m_1 g = \lambda \\ m_2\ddot{x}_2 - m_2 g = \lambda \\ x_1 + x_2 \qquad = \ell \end{cases}$$

Solve the system:

From the first two equations:

$$m_1\ddot{x}_1 - m_1 g = m_2\ddot{x}_2 - m_2 g$$

Using $\ddot{x}_2 = -\ddot{x}_1$ from the constraint:

$$m_1\ddot{x}_1 - m_1 g = -m_2\ddot{x}_1 - m_2 g$$

$$(m_1 + m_2)\ddot{x}_1 = (m_1 - m_2)g$$

$$\boxed{\ddot{x}_1 = \frac{m_1 - m_2}{m_1 + m_2}g, \quad \ddot{x}_2 = -\frac{m_1 - m_2}{m_1 + m_2}g}$$

Find Lagrange multiplier:

Substituting back:

$$\lambda = m_1\ddot{x}_1 - m_1 g = m_1\frac{m_1 - m_2}{m_1 + m_2}g - m_1 g = -\frac{2m_1 m_2}{m_1 + m_2}g$$

Generalized constraint forces:

The generalized constraint forces are:

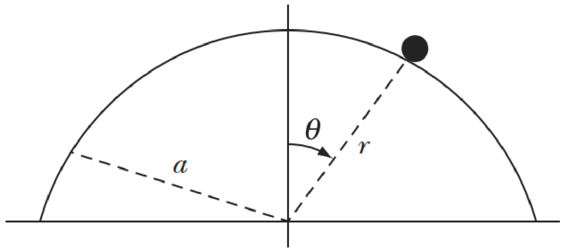
$$\begin{cases} Q_{x_1}^{\text{cons}} = \lambda \frac{\partial \psi}{\partial x_1} = -\frac{2m_1 m_2}{m_1 + m_2}g \\ Q_{x_2}^{\text{cons}} = \lambda \frac{\partial \psi}{\partial x_2} = -\frac{2m_1 m_2}{m_1 + m_2}g \end{cases}$$

These represent the tension forces in the string acting on each mass.

The magnitude of the tension is:

$$\boxed{T_{\text{string}} = \frac{2m_1 m_2}{m_1 + m_2}g}$$

Problem 8.8



A particle of mass m starts at rest on top of a smooth fixed hemisphere of radius a. Determine the angle at which the particle leaves the hemisphere from the Euler-Lagrange equation.

Solution

Set up the Lagrangian:

Using polar coordinates (r, θ) with origin at the center of the hemisphere:

$$L\left(r, \theta, \dot{r}, \dot{\theta}\right) = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - mgr\cos\theta$$

Constraint and Lagrange multiplier method:

Constraint: $\psi(r, \theta) = r - a = 0$

Modified Euler-Lagrange equations:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} &= \lambda\frac{\partial \psi}{\partial r} \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} &= \lambda\frac{\partial \psi}{\partial \theta} \\ \psi(r, \theta) &= 0 \end{cases}$$

Compute partial derivatives:

$$\begin{cases} \dfrac{\partial L}{\partial r} = mr\dot{\theta}^2 - mg\cos\theta \\ \dfrac{\partial L}{\partial \dot{r}} = m\dot{r} \\ \dfrac{\partial L}{\partial \theta} = mgr\sin\theta \\ \dfrac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \end{cases}$$

Also, $\frac{\partial \psi}{\partial r} = 1$ and $\frac{\partial \psi}{\partial \theta} = 0$.

Write the equations:

$$\begin{cases} m\ddot{r} - mr\dot{\theta}^2 + mg\cos\theta &= \lambda \\ mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} - mgr\sin\theta &= 0 \\ r - a &= 0 \end{cases}$$

Apply constraint:

On the hemisphere, $r = a$, $\dot{r} = 0$, $\ddot{r} = 0$. The equations become:

$$\begin{cases} -ma\dot{\theta}^2 + mg\cos\theta = \lambda \\ ma^2\ddot{\theta} - mga\sin\theta \;\; = 0 \end{cases}$$

Solve for $\dot{\theta}^2$:

From the second equation:

$$\ddot{\theta} = \frac{g}{a}\sin\theta$$

Using the chain rule $\ddot{\theta} = \dot{\theta}\frac{\mathrm{d}\dot{\theta}}{\mathrm{d}\theta}$:

$$\dot{\theta}\frac{\mathrm{d}\dot{\theta}}{\mathrm{d}\theta} = \frac{g}{a}\sin\theta$$

Integrating:

$$\int \dot{\theta}\,\mathrm{d}\dot{\theta} = \int \frac{g}{a}\sin\theta\,\mathrm{d}\theta$$

$$\frac{\dot{\theta}^2}{2} = -\frac{g}{a}\cos\theta + C$$

Apply initial conditions:

At $t = 0$: $\theta = 0$ and $\dot{\theta} = 0$ (starts from rest at the top):

$$0 = -\frac{g}{a} + C \quad \Rightarrow \quad C = \frac{g}{a}$$

Therefore:

$$\frac{\dot{\theta}^2}{2} = \frac{g}{a}(1 - \cos\theta)$$

$$\boxed{\dot{\theta}^2 = \frac{2g}{a}(1 - \cos\theta)}$$

Find constraint force:

Substituting into the first equation:

$$\lambda = -ma\dot{\theta}^2 + mg\cos\theta = -ma \cdot \frac{2g}{a}(1 - \cos\theta) + mg\cos\theta$$

$$\lambda = -2mg(1 - \cos\theta) + mg\cos\theta = -2mg + 2mg\cos\theta + mg\cos\theta$$

$$\lambda = mg(3\cos\theta - 2)$$

The constraint force (normal force) is:

$$Q_r^{\text{cons}} = \lambda\frac{\partial\psi}{\partial r} = \lambda = mg(3\cos\theta - 2)$$

Particle leaves when constraint force becomes zero:

The particle leaves the hemisphere when the normal force vanishes:

$$mg(3\cos\theta_0 - 2) = 0$$

$$3\cos\theta_0 = 2$$

$$\boxed{\theta_0 = \cos^{-1}\left(\frac{2}{3}\right) \approx 48.2^\circ}$$

Physical interpretation: At this angle, the required centripetal force exceeds what gravity can provide via the normal force, so the particle loses contact with the hemisphere and becomes a projectile.