Worksheet 2

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Classical Mechanics II

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Problem 1.2

Show that the acceleration of a particle moving along a trajectory $\boldsymbol{r}(t)$ is given by

$$a(t) = (\dot{v}(t))\hat{e}_T + \frac{v^2(t)}{\rho}\hat{e}_N$$

, where $\rho \equiv 1/\kappa$ is its radius of curvature.

Solution

Velocity vector:

The velocity vector is:

$$v(t) = \dot{\boldsymbol{r}}(t) = \dot{s}(t)\frac{\partial \boldsymbol{r}}{\partial s} = v(t)\hat{e}_T$$

Acceleration vector:

Taking the time derivative of the velocity:

$$\begin{aligned}
a(t) &= \dot{v}(t) \\
&= \dot{v}(t)\hat{e}_T + v(t)\dot{\hat{e}_T}(t) \\
&= \dot{v}(t)\hat{e}_T + v(t)\dot{s}(t)\frac{\partial \hat{e}_T}{\partial s} \\
&= \dot{v}(t)\hat{e}_T + v^2(t)\kappa\hat{e}_N \\
&= \dot{v}(t)\hat{e}_T + \frac{v^2(t)}{\rho}\hat{e}_N
\end{aligned}$$

where we used the definition of curvature $\kappa = |\frac{\partial \hat{e}_T}{\partial s}|$ and the fact that $\frac{\partial \hat{e}_T}{\partial s} = \kappa\hat{e}_N$, and $\rho = 1/\kappa$ is the radius of curvature.

Problem 1.3

Find the tangent, normal and binormal vectors, as well as curvature and torsion for the circular helix.

Solution

Position vector:

The position vector of the circular helix is:

$$\boldsymbol{r}(t) = a\cos(\omega t)\hat{e}_x + a\sin(\omega t)\hat{e}_y + b\omega t\hat{e}_z$$

where a, b, and ω are constants.

Velocity and speed:

$$\dot{\boldsymbol{r}}(t) = -a\omega\sin(\omega t)\hat{e}_x + a\omega\cos(\omega t)\hat{e}_y + b\omega\hat{e}_z$$

The arc length parameter:

$$s(t) = \int_0^t |\dot{\boldsymbol{r}}(\tau)|\,\mathrm{d}\tau = \omega\sqrt{a^2+b^2}t$$

Therefore:

$$\dot{s}(t) = \omega\sqrt{a^2+b^2}$$

Tangent vector:

$$\begin{aligned}\hat{e}_{T(t)} &= \frac{\partial \boldsymbol{r}(s)}{\partial s} = \frac{\dot{\boldsymbol{r}}(t)}{\dot{s}(t)} \\ &= \frac{1}{\sqrt{a^2+b^2}}\big(-a\sin(\omega t)\hat{e}_x + a\cos(\omega t)\hat{e}_y + b\hat{e}_z\big)\end{aligned}$$

Normal vector and curvature:

Taking the derivative of the tangent vector:

$$\dot{\hat{e}}_T(t) = \frac{a\omega}{\sqrt{a^2+b^2}}\big(-\cos(\omega t)\hat{e}_x - \sin(\omega t)\hat{e}_y\big)$$

Converting to arc length parameter:

$$\frac{\partial \hat{e}_{T(t)}}{\partial s} = \dot{\hat{e}}_T\frac{t}{\dot{s}}(t) = \frac{a}{a^2+b^2}\big(-\cos(\omega t)\hat{e}_x - \sin(\omega t)\hat{e}_y\big)$$

The curvature is:

$$\kappa(t) = |\frac{\partial \hat{e}_{T(t)}}{\partial s}| = \frac{a}{a^2+b^2}$$

The normal vector is:

$$\hat{e}_{N(t)} = \frac{1}{\kappa(t)}\frac{\partial \hat{e}_{T(t)}}{\partial s} = -\cos(\omega t)\hat{e}_x - \sin(\omega t)\hat{e}_y$$

Binormal vector:

$$\begin{aligned}\hat{e}_{B(t)} &= \hat{e}_{T(t)} \times \hat{e}_{N(t)} \\ &= \frac{1}{\sqrt{a^2+b^2}}\big(b\sin(\omega t)\hat{e}_x - b\cos(\omega t)\hat{e}_y + a\hat{e}_z\big)\end{aligned}$$

Torsion:

Taking the derivative of the binormal vector:

$$\dot{\hat{e}}_B(t) = \frac{b\omega}{\sqrt{a^2+b^2}}\big(\cos(\omega t)\hat{e}_x + \sin(\omega t)\hat{e}_y\big)$$

Converting to arc length parameter:

$$\frac{\partial \hat{e}_{B(t)}}{\partial s} = \dot{\hat{e}}_B \frac{t}{\dot{s}}(t) = \frac{b}{a^2+b^2}\big(\cos(\omega t)\hat{e}_x + \sin(\omega t)\hat{e}_y\big)$$

Using the Frenet-Serret formula $\frac{\partial \hat{e}_B}{\partial s} = -\tau \hat{e}_N$:

$$\tau(t) = -\hat{e}_{N(t)} \cdot \frac{\partial \hat{e}_{B(t)}}{\partial s} = \frac{b}{a^2+b^2}$$

Alternatively, we can compute:

$$\dot{\hat{e}}_N(t) = \omega\big(\sin(\omega t)\hat{e}_x - \cos(\omega t)\hat{e}_y\big)$$

Converting to arc length:

$$\frac{\partial \hat{e}_{N(t)}}{\partial s} = \dot{\hat{e}}_N \frac{t}{\dot{s}}(t) = \frac{1}{\sqrt{a^2+b^2}}\big(\sin(\omega t)\hat{e}_x - \cos(\omega t)\hat{e}_y\big)$$

Using the relationship $\tau = \hat{e}_B \cdot \frac{\partial \hat{e}_N}{\partial s}$:

$$\tau(t) = \hat{e}_{B(t)} \cdot \frac{\partial \hat{e}_{N(t)}}{\partial s} = \frac{b}{a^2+b^2}$$

Summary:

$$\begin{cases} \hat{e}_{T(t)} &= \frac{1}{\sqrt{a^2+b^2}}\big(-a\sin(\omega t)\hat{e}_x + a\cos(\omega t)\hat{e}_y + b\hat{e}_z\big) \\ \hat{e}_{N(t)} &= -\cos(\omega t)\hat{e}_x - \sin(\omega t)\hat{e}_y \\ \hat{e}_{B(t)} &= \frac{1}{\sqrt{a^2+b^2}}\big(b\sin(\omega t)\hat{e}_x - b\cos(\omega t)\hat{e}_y + a\hat{e}_z\big) \\ \kappa(t) &= \frac{a}{a^2+b^2} \\ \tau(t) &= \frac{b}{a^2+b^2} \end{cases}$$

Problem 1.4

Establish the relationship between unit basis vectors $\left(\hat{e}_\rho, \hat{e}_\varphi\right)$ of the polar coordinate system and the unit basis vectors $\left(\hat{e}_x, \hat{e}_y\right)$ of the Cartesian coordinate system.

Solution

Transformation from Cartesian to polar:

The position vector in Cartesian coordinates is:

$$\boldsymbol{r} = x\hat{e}_x + y\hat{e}_y = \rho\cos\varphi\hat{e}_x + \rho\sin\varphi\hat{e}_y$$

The unit basis vectors in polar coordinates are defined as:

$$\begin{cases} \hat{e}_\rho = \frac{\frac{\partial \boldsymbol{r}}{\partial \rho}}{\left|\frac{\partial \boldsymbol{r}}{\partial \rho}\right|} = \cos\varphi\hat{e}_x + \sin\varphi\hat{e}_y \\ \hat{e}_\varphi = \frac{\frac{\partial \boldsymbol{r}}{\partial \varphi}}{\left|\frac{\partial \boldsymbol{r}}{\partial \varphi}\right|} = -\sin\varphi\hat{e}_x + \cos\varphi\hat{e}_y \end{cases}$$

Inverse transformation:

To find the inverse transformation, we solve for $\left(\hat{e}_x, \hat{e}_y\right)$ in terms of $\left(\hat{e}_\rho, \hat{e}_\varphi\right)$. The transformation matrix has determinant $\cos^2\varphi + \sin^2\varphi = 1$, so its inverse is:

$$\begin{cases} \hat{e}_x = \cos\varphi\hat{e}_\rho - \sin\varphi\hat{e}_\varphi \\ \hat{e}_y = \sin\varphi\hat{e}_\rho + \cos\varphi\hat{e}_\varphi \end{cases}$$

Problem 1.5

Express the velocity and acceleration vectors in 2D polar coordinates.

Solution

Position vector:

In polar coordinates, the position vector is:

$$\boldsymbol{r}(t) = \rho(t)\hat{e}_\rho$$

Time derivatives of basis vectors:

From the relationship:

$$\begin{cases} \hat{e}_\rho &= \cos\varphi(t)\hat{e}_x + \sin\varphi(t)\hat{e}_y \\ \hat{e}_\varphi &= -\sin\varphi(t)\hat{e}_x + \cos\varphi(t)\hat{e}_y \end{cases}$$

Taking time derivatives:

$$\begin{cases} \dot{\hat{e}}_\rho(t) &= -\dot{\varphi}(t)\sin\varphi(t)\hat{e}_x + \dot{\varphi}(t)\cos\varphi(t)\hat{e}_y = \dot{\varphi}(t)\hat{e}_\varphi \\ \dot{\hat{e}}_\varphi(t) &= -\dot{\varphi}(t)\cos\varphi(t)\hat{e}_x - \dot{\varphi}(t)\sin\varphi(t)\hat{e}_y = -\dot{\varphi}(t)\hat{e}_\rho \end{cases}$$

Velocity vector:

Taking the time derivative of the position vector:

$$\begin{aligned} v(t) &= \dot{\boldsymbol{r}}(t) \\ &= \dot{\rho}(t)\hat{e}_\rho + \rho(t)\dot{\hat{e}}_\rho(t) \\ &= \dot{\rho}(t)\hat{e}_\rho + \rho(t)\dot{\varphi}(t)\hat{e}_\varphi \end{aligned}$$

Acceleration vector:

Taking the time derivative of the velocity:

$$\begin{aligned} a(t) &= \dot{v}(t) \\ &= \ddot{\rho}(t)\hat{e}_\rho + \dot{\rho}(t)\dot{\hat{e}}_\rho(t) + \dot{\rho}(t)\dot{\varphi}(t)\hat{e}_\varphi + \rho(t)\ddot{\varphi}(t)\hat{e}_\varphi + \rho(t)\dot{\varphi}(t)\dot{\hat{e}}_\varphi(t) \\ &= \ddot{\rho}(t)\hat{e}_\rho + \dot{\rho}(t)\dot{\varphi}(t)\hat{e}_\varphi + \dot{\rho}(t)\dot{\varphi}(t)\hat{e}_\varphi + \rho(t)\ddot{\varphi}(t)\hat{e}_\varphi - \rho(t)\dot{\varphi}^2(t)\hat{e}_\rho \\ &= [\ddot{\rho}(t) - \rho(t)\dot{\varphi}^2(t)]\hat{e}_\rho + [\rho(t)\ddot{\varphi}(t) + 2\dot{\rho}(t)\dot{\varphi}(t)]\hat{e}_\varphi \end{aligned}$$

Summary:

$$\begin{cases} v(t) = \dot{\rho}(t)\hat{e}_\rho + \rho(t)\dot{\varphi}(t)\hat{e}_\varphi \\ a(t) = [\ddot{\rho}(t) - \rho(t)\dot{\varphi}^2(t)]\hat{e}_\rho + [\rho(t)\ddot{\varphi}(t) + 2\dot{\rho}(t)\dot{\varphi}(t)]\hat{e}_\varphi \end{cases}$$

Problem 1.6

Express the spherical unit basis vectors $\left(\hat{e}_r, \hat{e}_\theta, \hat{e}_\varphi\right)$ in terms of Cartesian unit basis vectors $\left(\hat{e}_x, \hat{e}_y, \hat{e}_z\right)$.

Solution

Position vector in spherical coordinates:

The position vector in Cartesian coordinates is:

$$\boldsymbol{r} = x\hat{e}_x + y\hat{e}_y + z\hat{e}_z = r\sin\theta\cos\varphi\hat{e}_x + r\sin\theta\sin\varphi\hat{e}_y + r\cos\theta\hat{e}_z$$

Partial derivatives:

$$\begin{cases} \frac{\partial \boldsymbol{r}}{\partial r} = \sin\theta\cos\varphi\hat{e}_x + \sin\theta\sin\varphi\hat{e}_y + \cos\theta\hat{e}_z \\ \frac{\partial \boldsymbol{r}}{\partial \theta} = r\cos\theta\cos\varphi\hat{e}_x + r\cos\theta\sin\varphi\hat{e}_y - r\sin\theta\hat{e}_z \\ \frac{\partial \boldsymbol{r}}{\partial \varphi} = -r\sin\theta\sin\varphi\hat{e}_x + r\sin\theta\cos\varphi\hat{e}_y \end{cases}$$

Magnitudes:

$$\begin{cases} \left|\frac{\partial \boldsymbol{r}}{\partial r}\right| = \sqrt{\sin^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi + \cos^2\theta} = \sqrt{\sin^2\theta + \cos^2\theta} = 1 \\ \left|\frac{\partial \boldsymbol{r}}{\partial \theta}\right| = r\sqrt{\cos^2\theta\cos^2\varphi + \cos^2\theta\sin^2\varphi + \sin^2\theta} = r\sqrt{\cos^2\theta + \sin^2\theta} = r \\ \left|\frac{\partial \boldsymbol{r}}{\partial \varphi}\right| = r\sin\theta\sqrt{\sin^2\varphi + \cos^2\varphi} = r\sin\theta \end{cases}$$

Unit basis vectors:

$$\begin{cases} \hat{e}_r \equiv \frac{\frac{\partial \boldsymbol{r}}{\partial r}}{\left|\frac{\partial \boldsymbol{r}}{\partial r}\right|} = \sin\theta\cos\varphi\hat{e}_x + \sin\theta\sin\varphi\hat{e}_y + \cos\theta\hat{e}_z \\ \hat{e}_\theta \equiv \frac{\frac{\partial \boldsymbol{r}}{\partial \theta}}{\left|\frac{\partial \boldsymbol{r}}{\partial \theta}\right|} = \cos\theta\cos\varphi\hat{e}_x + \cos\theta\sin\varphi\hat{e}_y - \sin\theta\hat{e}_z \\ \hat{e}_\varphi \equiv \frac{\frac{\partial \boldsymbol{r}}{\partial \varphi}}{\left|\frac{\partial \boldsymbol{r}}{\partial \varphi}\right|} = -\sin\varphi\hat{e}_x + \cos\varphi\hat{e}_y \end{cases}$$

Verification of right-handed coordinate system:

The scalar triple product should equal 1:

$$\hat{e}_r \cdot \left(\hat{e}_\theta \times \hat{e}_\varphi\right) = 1$$

This can be verified by direct computation using the expressions above.

This confirms that the basis vectors form a right-handed orthonormal coordinate system.

Summary:

$$\begin{cases} \hat{e}_r = \sin\theta\cos\varphi\hat{e}_x + \sin\theta\sin\varphi\hat{e}_y + \cos\theta\hat{e}_z \\ \hat{e}_\theta = \cos\theta\cos\varphi\hat{e}_x + \cos\theta\sin\varphi\hat{e}_y - \sin\theta\hat{e}_z \\ \hat{e}_\varphi = -\sin\varphi\hat{e}_x + \cos\varphi\hat{e}_y \end{cases}$$