Worksheet 19

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Classical Mechanics II

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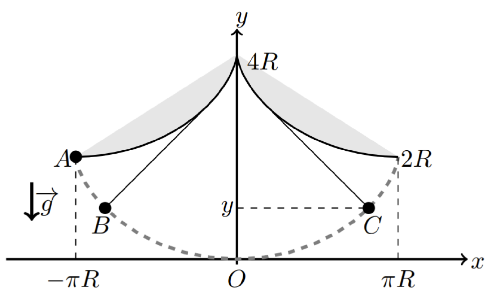
Problem 8.4



Huygens (1673) constructed a cycloidal pendulum with a point particle of mass m and a string of length $4R$ suspended from the cusp of an inverted cycloid.

Path of point mass (cycloid parametrization):

$$\begin{cases} x = R(\theta + \sin\theta), & -\pi \leq \theta \leq \pi \\ y = R(1 - \cos\theta) \end{cases}$$

Period: $T = 4\pi\sqrt{R/g}$ (independent of amplitude!)

Obtain the equation of motion for the cycloidal pendulum from the Euler-Lagrange equation.

Solution

Position vector:

The position vector of the particle is:

$$\boldsymbol{r}(t) = R[\theta(t) + \sin\theta(t)]\hat{e}_x + R[1 - \cos\theta(t)]\hat{e}_y$$

Velocity vector:

Taking the time derivative:

$$\dot{\boldsymbol{r}}(t) = R\dot{\theta}(t)[1 + \cos\theta(t)]\hat{e}_x + R\dot{\theta}(t)\sin\theta(t)\hat{e}_y$$

Kinetic energy:

$$\begin{aligned} T\left(\theta, \dot{\theta}, t\right) &= \frac{m}{2}\dot{\boldsymbol{r}}(t) \cdot \dot{\boldsymbol{r}}(t) \\ &= \frac{m}{2}R^2\dot{\theta}^2(t)\left[(1 + \cos\theta(t)^2 + \sin^2\theta(t)]\right) \\ &= \frac{m}{2}R^2\dot{\theta}^2(t)\left[1 + 2\cos\theta(t) + \cos^2\theta(t) + \sin^2\theta(t)\right] \\ &= \frac{m}{2}R^2\dot{\theta}^2(t)[2 + 2\cos\theta(t)] \\ &= 2mR^2\dot{\theta}^2(t)\cos^2\left(\frac{\theta(t)}{2}\right) \end{aligned}$$

where we used the identity $1 + \cos\theta = 2\cos^2(\theta/2)$.

Potential energy:

$$U(\theta) = mgy(t) = mgR[1 - \cos\theta(t)] = 2mgR\sin^2\left(\frac{\theta(t)}{2}\right)$$

where we used $1 - \cos\theta = 2\sin^2(\theta/2)$.

Lagrangian:

$$L\big(\theta, \dot{\theta}, t\big) = T - U = 2mR^2\dot{\theta}^2(t)\cos^2\left(\frac{\theta(t)}{2}\right) - 2mgR\sin^2\left(\frac{\theta(t)}{2}\right)$$

Change to arc length coordinate:

The arc length along the cycloid is:

$$s(t) = 4R\sin\left(\frac{\theta(t)}{2}\right)$$

Taking the time derivative:

$$\dot{s}(t) = 2R\dot{\theta}(t)\cos\left(\frac{\theta(t)}{2}\right)$$

From this, we can express:

$$\dot{\theta}(t)\cos\left(\frac{\theta(t)}{2}\right) = \frac{\dot{s}(t)}{2R}$$

Express Lagrangian in terms of s:

Substituting into the kinetic energy:

$$T = 2mR^2\dot{\theta}^2(t)\cos^2\left(\frac{\theta(t)}{2}\right) = 2mR^2\left[\left(\dot{\theta}(t)\cos\left(\frac{\theta(t)}{2}\right)\right)\right]^2 = 2mR^2\left(\frac{\dot{s}(t)}{2R}\right)^2 = \frac{1}{2}m\dot{s}^2(t)$$

For the potential energy, using $\sin(\theta/2) = s/(4R)$:

$$U = 2mgR\sin^2\left(\frac{\theta(t)}{2}\right) = 2mgR\left(\frac{s(t)}{4R}\right)^2 = \frac{mg}{8R}s^2(t)$$

The Lagrangian becomes:

$$L(s, \dot{s}, t) = \frac{1}{2}m\dot{s}^2(t) - \frac{mg}{8R}s^2(t)$$

Apply Euler-Lagrange equation:

$$\frac{\partial L}{\partial s} = -\frac{mg}{4R}s(t), \quad \frac{\partial L}{\partial \dot{s}} = m\dot{s}(t)$$

Euler-Lagrange equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{s}}\right) - \frac{\partial L}{\partial s} = 0$$

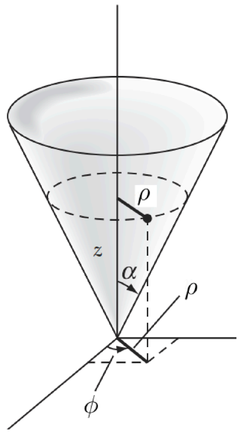
$$m\ddot{s}(t) + \frac{mg}{4R}s(t) = 0$$

$$\boxed{\ddot{s}(t) + \frac{g}{4R}s(t) = 0 \equiv \ddot{s}(t) + \omega^2 s(t) = 0}$$

where $\omega = \sqrt{g/(4R)}$.

Conclusion: This is simple harmonic motion in the variable s, with period $T = 2\pi/\omega = 4\pi\sqrt{R/g}$, independent of amplitude. This is the key property of the cycloidal pendulum—it is an isochronous oscillator.

Problem 8.5



A particle of mass m is constrained to move on the inside surface of a smooth cone of half-angle α. The particle is subjected to gravitational force.

Express the Lagrangian in suitable generalized coordinates and obtain the equations of motion.

Solution

Choose generalized coordinates:

We use cylindrical coordinates (ρ, φ, z) with the cone constraint. The cone surface is described by:

$$z = \rho \cot \alpha$$

Since the particle is constrained to the cone, we have two degrees of freedom: (ρ, φ).

Coordinate transformation:

The Cartesian coordinates in terms of generalized coordinates:

$$\begin{cases} x = \rho \cos \varphi \\ y = \rho \sin \varphi \\ z = \rho \cot \alpha \end{cases}$$

Velocities:

Taking time derivatives:

$$\begin{cases} \dot{x} = \dot{\rho} \cos \varphi - \rho \dot{\varphi} \sin \varphi \\ \dot{y} = \dot{\rho} \sin \varphi + \rho \dot{\varphi} \cos \varphi \\ \dot{z} = \dot{\rho} \cot \alpha \end{cases}$$

Kinetic energy:

$$\begin{aligned} T &= \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \\ &= \frac{m}{2}[(\dot{\rho}^2 \cos^2 \varphi - 2\rho\dot{\rho}\dot{\varphi} \cos \varphi \sin \varphi + \rho^2 \dot{\varphi}^2 \sin^2 \varphi) \\ &\quad + (\dot{\rho}^2 \sin^2 \varphi + 2\rho\dot{\rho}\dot{\varphi} \sin \varphi \cos \varphi + \rho^2 \dot{\varphi}^2 \cos^2 \varphi) + \dot{\rho}^2 \cot^2 \alpha] \\ &= \frac{m}{2}[\dot{\rho}^2(\cos^2 \varphi + \sin^2 \varphi + \cot^2 \alpha) + \rho^2 \dot{\varphi}^2 (\sin^2 \varphi + \cos^2 \varphi)] \\ &= \frac{m}{2}[\dot{\rho}^2(1 + \cot^2 \alpha) + \rho^2 \dot{\varphi}^2] \\ &= \frac{m}{2}[\dot{\rho}^2 \csc^2 \alpha + \rho^2 \dot{\varphi}^2] \end{aligned}$$

where we used $1 + \cot^2 \alpha = \csc^2 \alpha$.

Potential energy:

$$U(\rho) = mgz = mg\rho\cot\alpha$$

Lagrangian:

$$L(\rho, \varphi, \dot{\rho}, \dot{\varphi}) = \frac{m}{2}(\dot{\rho}^2\csc^2\alpha + \rho^2\dot{\varphi}^2) - mg\rho\cot\alpha$$

Euler-Lagrange equation for φ:

Note that φ is a cyclic coordinate ($\frac{\partial L}{\partial \varphi} = 0$):

$$\frac{\partial L}{\partial \varphi} = 0, \quad \frac{\partial L}{\partial \dot{\varphi}} = m\rho^2\dot{\varphi}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) - \frac{\partial L}{\partial \varphi} = 0$$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}(m\rho^2\dot{\varphi}) = 0$$

$$\boxed{m\rho^2(t)\dot{\varphi}(t) = \text{constant} \equiv \ell}$$

This is the conservation of angular momentum about the vertical axis.

Euler-Lagrange equation for ρ:

$$\frac{\partial L}{\partial \rho} = m\rho\dot{\varphi}^2 - mg\cot\alpha, \quad \frac{\partial L}{\partial \dot{\rho}} = m\dot{\rho}\csc^2\alpha$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\rho}}\right) - \frac{\partial L}{\partial \rho} = 0$$

$$m\ddot{\rho}\csc^2\alpha - m\rho\dot{\varphi}^2 + mg\cot\alpha = 0$$

Dividing by m and rearranging:

$$\boxed{\ddot{\rho}(t) = \rho(t)\dot{\varphi}^2(t)\sin^2\alpha - g\sin\alpha\cos\alpha}$$

Alternative form: Using the conserved angular momentum $\ell = \rho^2\dot{\varphi}$, we have $\dot{\varphi} = \ell/\rho^2$:

$$\ddot{\rho}(t) = \frac{\ell^2}{\rho^3(t)}\sin^2\alpha - g\sin\alpha\cos\alpha$$

This can be written as motion in an effective potential:

$$\ddot{\rho}(t) = -\frac{\partial U_{\text{eff}}}{\partial \rho}$$

where the effective potential is:

$$\boxed{U_{\text{eff}}(\rho) = \frac{\ell^2}{2m\rho^2}\csc^2\alpha + mg\rho\cot\alpha}$$

The first term is the centrifugal potential, and the second is the gravitational potential along the cone.