Worksheet 18

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Classical Mechanics II

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Problem 8.3

Show that the Euler-Lagrange equation of motion is covariant under point transformation.

Solution

Consider a point transformation:

$$\bar{q}_j = \bar{q}_j(\{q_i\}, t) \quad \Rightarrow \quad \mathcal{L}\big(\{\bar{q}_j, \dot{\bar{q}}_j\}, t\big) = \mathcal{L}\big(\bar{q}_j(\{q_i\}, t), \dot{\bar{q}}_j(\{q_i, \dot{q}_i\}, t), t\big) = \mathcal{L}(\{q_i, \dot{q}_i\}, t)$$

Partial derivative with respect to $\bar{q}_k$:

$$\frac{\partial \mathcal{L}}{\partial \bar{q}_k} = \sum_{j=1}^{M} \left[\frac{\partial \mathcal{L}}{\partial \bar{q}_j} \frac{\partial \bar{q}_j}{\partial \bar{q}_k} + \frac{\partial \mathcal{L}}{\partial \dot{\bar{q}}_j} \frac{\partial \dot{\bar{q}}_j}{\partial \bar{q}_k} \right]$$

Partial derivative with respect to $\dot{\bar{q}}_k$:

$$\frac{\partial \mathcal{L}}{\partial \dot{\bar{q}}_k} = \sum_{j=1}^{M} \frac{\partial \mathcal{L}}{\partial \dot{\bar{q}}_j} \frac{\partial \dot{\bar{q}}_j}{\partial \dot{\bar{q}}_k} = \sum_{j=1}^{M} \frac{\partial \mathcal{L}}{\partial \dot{\bar{q}}_j} \frac{\partial \bar{q}_j}{\partial \bar{q}_k}$$

Time derivative:

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \mathcal{L}}{\partial \dot{\bar{q}}_k} \right) = \sum_{j=1}^{M} \left[\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \mathcal{L}}{\partial \dot{\bar{q}}_j} \right) \frac{\partial \bar{q}_j}{\partial \bar{q}_k} + \frac{\partial \mathcal{L}}{\partial \dot{\bar{q}}_j} \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \bar{q}_j}{\partial \bar{q}_k} \right) \right]$$

Euler-Lagrange equation in barred coordinates:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \mathcal{L}}{\partial \dot{\bar{q}}_k} \right) - \frac{\partial \mathcal{L}}{\partial \bar{q}_k} &= \sum_{j=1}^{M} \left[\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \mathcal{L}}{\partial \dot{\bar{q}}_j} \right) \frac{\partial \bar{q}_j}{\partial \bar{q}_k} + \frac{\partial \mathcal{L}}{\partial \dot{\bar{q}}_j} \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \bar{q}_j}{\partial \bar{q}_k} \right) - \frac{\partial \mathcal{L}}{\partial \bar{q}_j} \frac{\partial \bar{q}_j}{\partial \bar{q}_k} - \frac{\partial \mathcal{L}}{\partial \dot{\bar{q}}_j} \frac{\partial \dot{\bar{q}}_j}{\partial \bar{q}_k} \right] \\
&= \sum_{j=1}^{M} \left\{ \frac{\partial \bar{q}_j}{\partial \bar{q}_k} \left[\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \mathcal{L}}{\partial \dot{\bar{q}}_j} \right) - \frac{\partial \mathcal{L}}{\partial \bar{q}_j} \right] + \frac{\partial \mathcal{L}}{\partial \dot{\bar{q}}_j} \left[\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \bar{q}_j}{\partial \bar{q}_k} \right) - \frac{\partial \dot{\bar{q}}_j}{\partial \bar{q}_k} \right] \right\} \\
&= 0
\end{aligned}$$

where we used the Euler-Lagrange equation in unbarred coordinates:

$$\frac{\partial \mathcal{L}}{\partial \bar{q}_j} - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \mathcal{L}}{\partial \dot{\bar{q}}_j} \right) = 0$$

Verification of the second term:

From $\bar{q}_j = \bar{q}_j(\{q_i\}, t)$:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \bar{q}_j}{\partial \bar{q}_k}\right) &= \sum_{i=1}^{M} \frac{\partial}{\partial q_i}\left(\frac{\partial \bar{q}_j}{\partial \bar{q}_k}\right)\dot{q}_i + \frac{\partial}{\partial t}\left(\frac{\partial \bar{q}_j}{\partial \bar{q}_k}\right) \\
&= \sum_{i=1}^{M} \frac{\partial}{\partial \bar{q}_k}\left(\frac{\partial \bar{q}_j}{\partial q_i}\right)\dot{q}_i + \frac{\partial}{\partial \bar{q}_k}\left(\frac{\partial \bar{q}_j}{\partial t}\right) \\
&= \sum_{i=1}^{M} \frac{\partial}{\partial \bar{q}_k}\left(\frac{\partial \bar{q}_j}{\partial q_i}\dot{q}_i\right) + \frac{\partial}{\partial \bar{q}_k}\left(\frac{\partial \bar{q}_j}{\partial t}\right) \\
&= \frac{\partial}{\partial \bar{q}_k}\left(\sum_{i=1}^{M} \frac{\partial \bar{q}_j}{\partial q_i}\dot{q}_i + \frac{\partial \bar{q}_j}{\partial t}\right) \\
&= \frac{\partial}{\partial \bar{q}_k}\left(\frac{\mathrm{d}\bar{q}_j}{\mathrm{d}t}\right) = \frac{\partial \dot{\bar{q}}_j}{\partial \bar{q}_k}
\end{aligned}$$

Therefore, the second term vanishes:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \bar{q}_j}{\partial \bar{q}_k}\right) - \frac{\partial \dot{\bar{q}}_j}{\partial \bar{q}_k} = 0$$

Hence, the Euler-Lagrange equation is covariant under point transformation:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\bar{q}}_k}\right) - \frac{\partial \mathcal{L}}{\partial \bar{q}_k} = 0}$$