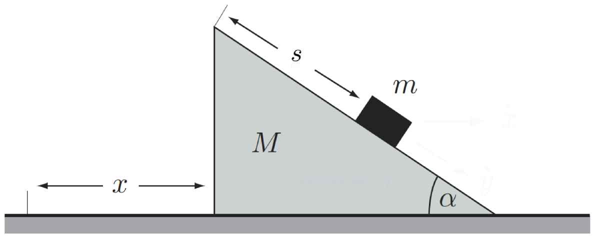
Worksheet 17

Parth Bhargava · A0310667E

PC3261
Classical Mechanics II

November 21, 2025

Problem 8.1



A block of mass m is free to slide on the wedge of mass M which can slide on the horizontal table, both with negligible friction.

Generalized coordinates: s is the distance of the block from the top of the wedge and x is the distance of the wedge from any convenient fixed point on the table.

Find the acceleration of the wedge, and acceleration of the block relative to the wedge from Lagrange's equation.

Solution

Position vectors:

$$\begin{cases} \boldsymbol{r}_1(t) = x(t)\hat{e}_x \\ \boldsymbol{r}_2(t) = [x(t) + s(t)\cos\alpha]\hat{e}_x + [H - s(t)\sin\alpha]\hat{e}_y \end{cases}$$

Kinetic energy:

$$\begin{aligned} T \equiv T(x, s, \dot{x}, \dot{s}, t) &= \frac{M}{2}\dot{\boldsymbol{r}}_1(t) \cdot \dot{\boldsymbol{r}}_1(t) + \frac{m}{2}\dot{\boldsymbol{r}}_2(t) \cdot \dot{\boldsymbol{r}}_2(t) \\ &= \frac{M}{2}\dot{x}^2(t) + \frac{m}{2}[\dot{s}^2(t) + 2\dot{s}(t)\dot{x}(t)\cos\alpha + \dot{x}^2(t)] \end{aligned}$$

Potential energy:

$$U \equiv U(x, s) = mgy_2(t) = mgH - mgs(t)\sin\alpha$$

Lagrange's equation for $x(t)$:

$$\begin{cases} \dfrac{\partial T}{\partial \dot{x}} = (M + m)\dot{x}(t) + m\dot{s}(t)\cos\alpha \\ \dfrac{\partial T}{\partial x} = 0 \\ \dfrac{\partial U}{\partial x} = 0 \end{cases}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} = -\frac{\partial U}{\partial x} \quad \Rightarrow \quad (M + m)\ddot{x}(t) + m\ddot{s}(t)\cos\alpha = \text{constant}$$

Lagrange's equation for $s(t)$:

$$\begin{cases} \dfrac{\partial T}{\partial \dot{s}} = m\dot{s}(t) + m\dot{x}(t)\cos\alpha \\ \dfrac{\partial T}{\partial s} = 0 \\ \dfrac{\partial U}{\partial s} = -mg\sin\alpha \end{cases}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{s}}\right) - \frac{\partial T}{\partial s} = -\frac{\partial U}{\partial s} \quad \Rightarrow \quad m\ddot{s}(t) + m\ddot{x}(t)\cos\alpha = mg\sin\alpha$$

System of equations:

$$\begin{cases} (M+m)\ddot{x}(t) + m\ddot{s}(t)\cos\alpha = \text{constant} \\ m\ddot{s}(t) + m\ddot{x}(t)\cos\alpha = mg\sin\alpha \end{cases}$$

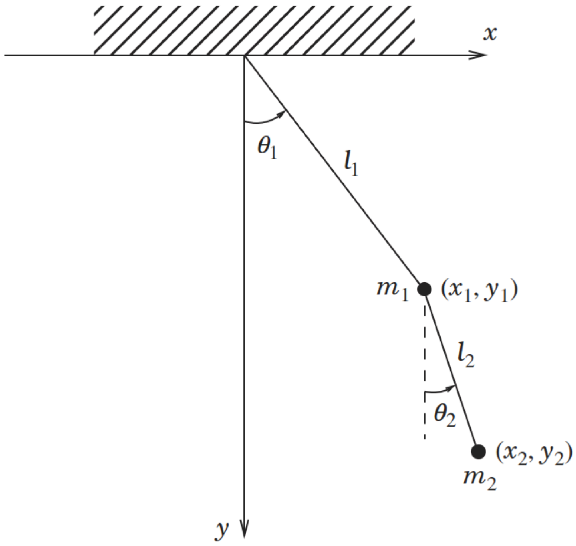
Assuming the system starts from rest (constant = 0):

$$\begin{cases} (M+m)\ddot{x}(t) + m\ddot{s}(t)\cos\alpha = 0 \\ m\ddot{s}(t) + m\ddot{x}(t)\cos\alpha = mg\sin\alpha \end{cases}$$

Solving for accelerations:

$$\boxed{\begin{cases} \ddot{x}(t) = -\dfrac{mg\sin\alpha\cos\alpha}{M + m\sin^2\alpha} \\ \ddot{s}(t) = \dfrac{(M+m)g\sin\alpha}{M + m\sin^2\alpha} \end{cases}}$$

Problem 8.2



A plane double pendulum consists of two light and inextensible rods of lengths ℓ_1 and ℓ_2 respectively. Two point masses, m_1 and m_2, are respectively attached at the end of each rod.

Holonomic constraints:

$$\begin{cases} f_1 = x_1^2 + y_1^2 - \ell_1^2 = 0 \\ f_2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 - \ell_2^2 = 0 \end{cases}$$

Generalized coordinates: $(q_1, q_2) \equiv (\theta_1, \theta_2)$

Obtain the equations of motion for the plane double pendulum from the Euler Lagrange equation.

Solution

Holonomic constraints:

$$\begin{cases} f_1 = x_1^2 + y_1^2 - \ell_1^2 = 0 \\ f_2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 - \ell_2^2 = 0 \end{cases}$$

Position vectors:

$$\begin{cases} \boldsymbol{r}_1(t) = \ell_1 \sin\theta_1(t)\hat{e}_x + \ell_1 \cos\theta_1(t)\hat{e}_y \\ \boldsymbol{r}_2(t) = [\ell_1 \sin\theta_1(t) + \ell_2 \sin\theta_2(t)]\hat{e}_x + [\ell_1 \cos\theta_1(t) + \ell_2 \cos\theta_2(t)]\hat{e}_y \end{cases}$$

Kinetic energy:

$$\begin{aligned} T \equiv T\big(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2, t\big) &= \frac{m_1}{2}\dot{\boldsymbol{r}}_1(t) \cdot \dot{\boldsymbol{r}}_1(t) + \frac{m_2}{2}\dot{\boldsymbol{r}}_2(t) \cdot \dot{\boldsymbol{r}}_2(t) \\ &= \frac{m_1 + m_2}{2}\ell_1^2\dot{\theta}_1^2(t) + \frac{m_2}{2}\ell_2^2\dot{\theta}_2^2(t) + m_2\ell_1\ell_2\dot{\theta}_1(t)\dot{\theta}_2(t)\cos[\theta_1(t) - \theta_2(t)] \end{aligned}$$

Potential energy:

$$\begin{aligned} U \equiv U(\theta_1, \theta_2) &= -m_1 g y_1(t) - m_2 g y_2(t) \\ &= -(m_1 + m_2)g\ell_1 \cos\theta_1(t) - m_2 g\ell_2 \cos\theta_2(t) \end{aligned}$$

Lagrangian:

$$\mathcal{L} \equiv \mathcal{L}\big(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2, t\big) = T - U$$

$$\begin{aligned} = \frac{m_1 + m_2}{2}\ell_1^2\dot{\theta}_1^2(t) + \frac{m_2}{2}\ell_2^2\dot{\theta}_2^2(t) + m_2\ell_1\ell_2\dot{\theta}_1(t)\dot{\theta}_2(t)\cos[\theta_1(t) - \theta_2(t)] \\ +(m_1 + m_2)g\ell_1 \cos\theta_1(t) + m_2 g\ell_2 \cos\theta_2(t) \end{aligned}$$

Euler-Lagrange equation for $\boldsymbol{\theta_1(t)}$:

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial \dot{\theta}_1} = (m_1 + m_2)\ell_1^2\dot{\theta}_1(t) + m_2\ell_1\ell_2\dot{\theta}_2(t)\cos[\theta_1(t) - \theta_2(t)] \\ \dfrac{\partial \mathcal{L}}{\partial \theta_1} = -m_2\ell_1\ell_2\dot{\theta}_1(t)\dot{\theta}_2(t)\sin[\theta_1(t) - \theta_2(t)] - (m_1 + m_2)g\ell_1 \sin\theta_1(t) \end{cases}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right) - \frac{\partial \mathcal{L}}{\partial \theta_1} = 0$$

$$\boxed{\begin{gathered}(m_1 + m_2)\ell_1^2\ddot{\theta}_1(t) + m_2\ell_1\ell_2\ddot{\theta}_2(t)\cos[\theta_1(t) - \theta_2(t)] \\ +m_2\ell_1\ell_2\dot{\theta}_2^2(t)\sin[\theta_1(t) - \theta_2(t)] + (m_1 + m_2)g\ell_1\sin\theta_1(t) = 0\end{gathered}}$$

Euler-Lagrange equation for $\boldsymbol{\theta_2(t)}$:

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial \dot{\theta}_2} = m_2\ell_2^2\dot{\theta}_2(t) + m_2\ell_1\ell_2\dot{\theta}_1(t)\cos[\theta_1(t) - \theta_2(t)] \\ \dfrac{\partial \mathcal{L}}{\partial \theta_2} = m_2\ell_1\ell_2\dot{\theta}_1(t)\dot{\theta}_2(t)\sin[\theta_1(t) - \theta_2(t)] - m_2 g\ell_2\sin\theta_2(t) \end{cases}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}\right) - \frac{\partial \mathcal{L}}{\partial \theta_2} = 0$$

$$\boxed{m_2\ell_2^2\ddot{\theta}_2(t) + m_2\ell_1\ell_2\ddot{\theta}_1(t)\cos[\theta_1(t) - \theta_2(t)] - m_2\ell_1\ell_2\dot{\theta}_1^2(t)\sin[\theta_1(t) - \theta_2(t)] + m_2 g\ell_2\sin\theta_2(t) = 0}$$