# Worksheet 16

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**PC3261**
Classical Mechanics II

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**Problem 7.4**

Obtain the expression for the inertial force term in the d'Alembert's principle.

**Solution**

Starting with the position vector as a function of generalized coordinates:

$$\boldsymbol{r}_\alpha = \boldsymbol{r}_\alpha(\{q_k(t)\}, t)$$

Taking the total time derivative:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \boldsymbol{r}_\alpha}{\partial q_k}\right) &= \sum_{i=1}^{M} \frac{\partial}{\partial q_i}\left(\frac{\partial \boldsymbol{r}_\alpha}{\partial q_k}\right)\dot{q}_i + \frac{\partial}{\partial t}\left(\frac{\partial \boldsymbol{r}_\alpha}{\partial q_k}\right) \\
&= \sum_{i=1}^{M} \frac{\partial}{\partial q_k}\left(\frac{\partial \boldsymbol{r}_\alpha}{\partial q_i}\right)\dot{q}_i + \frac{\partial}{\partial q_k}\left(\frac{\partial \boldsymbol{r}_\alpha}{\partial t}\right) \\
&= \sum_{i=1}^{M} \frac{\partial}{\partial q_k}\left(\frac{\partial \boldsymbol{r}_\alpha}{\partial q_i}\dot{q}_i\right) + \frac{\partial}{\partial q_k}\left(\frac{\partial \boldsymbol{r}_\alpha}{\partial t}\right) \\
&= \frac{\partial}{\partial q_k}\left[\sum_{i=1}^{M}\left(\frac{\partial \boldsymbol{r}_\alpha}{\partial q_i}\dot{q}_i\right) + \frac{\partial \boldsymbol{r}_\alpha}{\partial t}\right] \\
&= \frac{\partial \dot{\boldsymbol{r}}_\alpha}{\partial q_k}
\end{aligned}$$

For the virtual displacement:

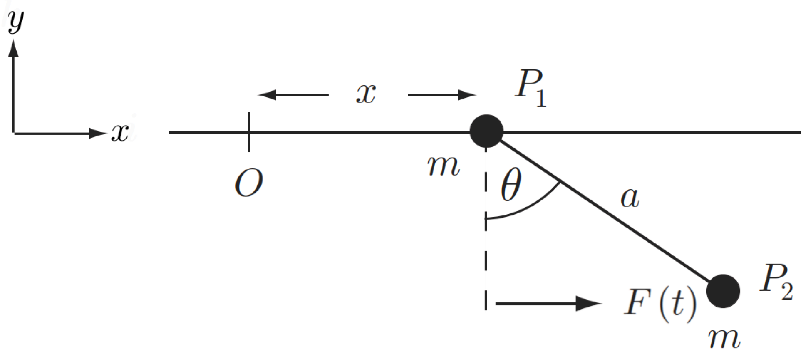
$$\boldsymbol{r}_\alpha = \boldsymbol{r}_\alpha(\{q_k(t)\}, t) \quad \Rightarrow \quad \delta \boldsymbol{r}_\alpha = \sum_{k=1}^{M} \frac{\partial \boldsymbol{r}_\alpha}{\partial q_k}\delta q_k$$

The inertial force term in d'Alembert's principle:

$$\begin{aligned}
-\sum_{\alpha=1}^{N} m_\alpha \ddot{\boldsymbol{r}}_\alpha \cdot \delta \boldsymbol{r}_\alpha &= -\sum_{\alpha=1}^{N}\sum_{k=1}^{M} m_\alpha \ddot{\boldsymbol{r}}_\alpha \cdot \frac{\partial \boldsymbol{r}_\alpha}{\partial q_k}\delta q_k \\
&= -\sum_{k=1}^{M}\sum_{\alpha=1}^{N}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(m_\alpha \ddot{\boldsymbol{r}}_\alpha \cdot \frac{\partial \boldsymbol{r}_\alpha}{\partial q_k}\right) - m_\alpha \ddot{\boldsymbol{r}}_\alpha \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \boldsymbol{r}_\alpha}{\partial q_k}\right)\right]\delta q_k \\
&= -\sum_{k=1}^{M}\sum_{\alpha=1}^{N}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(m_\alpha \ddot{\boldsymbol{r}}_\alpha \cdot \frac{\partial \boldsymbol{r}_\alpha}{\partial q_k}\right) - m_\alpha \ddot{\boldsymbol{r}}_\alpha \cdot \frac{\partial \dot{\boldsymbol{r}}_\alpha}{\partial q_k}\right]\delta q_k \\
&= -\sum_{k=1}^{M}\sum_{\alpha=1}^{N}\left\{\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial}{\partial \dot{q}_k}\left(\frac{1}{2}m_\alpha \dot{\boldsymbol{r}}_\alpha \cdot \dot{\boldsymbol{r}}_\alpha\right)\right] - \frac{\partial}{\partial q_k}\left(\frac{1}{2}m_\alpha \dot{\boldsymbol{r}}_\alpha \cdot \dot{\boldsymbol{r}}_\alpha\right)\right\}\delta q_k
\end{aligned}$$

**Problem 7.5**

Two identical particles, $P_1$ and $P_2$, with mass $m$ are connected by a light rigid rod of length $a$. $P_1$ is constrained to move along a fixed horizontal frictionless rail and the system moves in the vertical plane through the rail. An external force $F(t)\hat{e}_x$ is acted on $P_2$.




**Generalized coordinates:** $(q_1, q_2) \equiv (x, \theta)$

Use Lagrange's equation to obtain equations of motions for $x(t)$ and $\theta(t)$.

**Solution**

**Position vectors:**

$$\begin{cases} \boldsymbol{r}_1(t) = x(t)\hat{e}_x \\ \boldsymbol{r}_2(t) = [x(t) + a\sin\theta(t)]\hat{e}_x - a\cos\theta(t)\hat{e}_y \end{cases}$$

**Velocity vectors:**

$$\begin{cases} \dot{\boldsymbol{r}}_1(t) = \dot{x}(t)\hat{e}_x \\ \dot{\boldsymbol{r}}_2(t) = \left[\dot{x}(t) + a\dot{\theta}(t)\cos\theta(t)\right]\hat{e}_x + a\dot{\theta}(t)\sin\theta(t)\hat{e}_y \end{cases}$$

**Kinetic energy:**

$$\begin{aligned} T \equiv T\big(x, \theta, \dot{x}, \dot{\theta}, t\big) &= \frac{m}{2}\dot{\boldsymbol{r}}_1(t)\cdot\dot{\boldsymbol{r}}_1(t) + \frac{m}{2}\dot{\boldsymbol{r}}_2(t)\cdot\dot{\boldsymbol{r}}_2(t) \\ &= \frac{1}{2}m\dot{x}^2(t) + ma^2\dot{\theta}^2(t) + ma\dot{x}(t)\dot{\theta}(t)\cos\theta(t) \end{aligned}$$

**Applied forces and generalized forces:**

$$\begin{cases} \boldsymbol{F}_1(t) = -mg\hat{e}_y \\ \boldsymbol{F}_2(t) = -mg\hat{e}_y + F(t)\hat{e}_x \end{cases}$$

$$\begin{cases} Q_x(t) = \boldsymbol{F}_1(t)\cdot\frac{\partial\boldsymbol{r}_1}{\partial x} + \boldsymbol{F}_2(t)\cdot\frac{\partial\boldsymbol{r}_2}{\partial x} = F(t) \\ Q_\theta(t) = \boldsymbol{F}_1(t)\cdot\frac{\partial\boldsymbol{r}_1}{\partial\theta} + \boldsymbol{F}_2(t)\cdot\frac{\partial\boldsymbol{r}_2}{\partial\theta} = a\cos\theta(t)F(t) - mga\sin\theta(t) \end{cases}$$

**Lagrange's equation for $x(t)$:**

$$\begin{cases} \dfrac{\partial T}{\partial x} = 0 \\ \dfrac{\partial T}{\partial \dot{x}} = 2m\dot{x}(t) + ma\dot{\theta}(t)\cos\theta(t) \end{cases}$$

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} = \mathcal{Q}_x(t)$$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{dt}}\Big[2m\dot{x}(t) + ma\dot{\theta}(t)\cos\theta(t)\Big] = F(t)$$

$$\boxed{2m\ddot{x}(t) + ma\cos\theta(t)\ddot{\theta}(t) - ma\sin\theta(t)\dot{\theta}^2(t) = F(t)}$$

**Lagrange's equation for $\theta(t)$:**

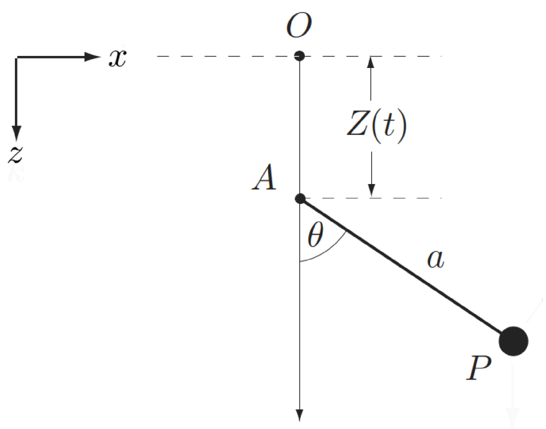
$$\begin{cases} \dfrac{\partial T}{\partial \theta} = -ma\sin\theta(t)\dot{x}(t)\dot{\theta}(t) \\[2ex] \dfrac{\partial T}{\partial \dot{\theta}} = ma^2\dot{\theta}(t) + ma\cos\theta(t)\dot{x}(t) \end{cases}$$

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = \mathcal{Q}_\theta(t)$$

$$\frac{\mathrm{d}}{\mathrm{dt}}\Big[ma^2\dot{\theta}(t) + ma\cos\theta(t)\dot{x}(t)\Big] + ma\sin\theta(t)\dot{x}(t)\dot{\theta}(t) = a\cos\theta(t)F(t) - mga\sin\theta(t)$$

$$\boxed{ma^2\ddot{\theta}(t) + ma\cos\theta(t)\ddot{x}(t) = a\cos\theta(t)F(t) - mga\sin\theta(t)}$$

**Problem 7.6**




A simple pendulum in which the pivot is made to move vertically so that its distance from the fixed origin at time $t$ is $Z(t) = Z_0\cos\Omega t$. The string is a light rigid rod of length $a$ that cannot go slack.

**Generalized coordinate:** $q_1 \equiv \theta$

Use Lagrange's equation to obtain equations of motion for $\theta(t)$.

**Solution**

**Position vector:**

$$\boldsymbol{r}(t) = a\sin\theta(t)\hat{e}_x + [Z(t) + a\cos\theta(t)]\hat{e}_z$$

**Velocity vector:**

$$\dot{\boldsymbol{r}}(t) = a\dot{\theta}(t)\cos\theta(t)\hat{e}_x + \Big[\dot{Z}(t) - a\dot{\theta}(t)\sin\theta(t)\Big]\hat{e}_z$$

**Kinetic energy:**

$$\begin{aligned} T \equiv T\big(\theta, \dot{\theta}, t\big) &= \frac{m}{2}\dot{\boldsymbol{r}}(t)\cdot\dot{\boldsymbol{r}}(t) \\ &= \frac{m}{2}\Big[a^2\dot{\theta}^2(t) + \dot{Z}^2(t) - 2a\dot{Z}(t)\dot{\theta}(t)\sin\theta(t)\Big] \end{aligned}$$

**Partial derivatives:**

$$\frac{\partial T}{\partial \theta} = -ma\dot{Z}(t)\dot{\theta}(t)\cos\theta(t), \quad \frac{\partial T}{\partial \dot{\theta}} = ma^2\dot{\theta}(t) - ma\dot{Z}(t)\sin\theta(t)$$

**Applied force and generalized force:**

$$\boldsymbol{F}(t) = mg\hat{e}_z \quad \Rightarrow \quad Q_\theta(t) = \boldsymbol{F}(t)\cdot\frac{\partial \boldsymbol{r}}{\partial \theta} = -mga\sin\theta(t)$$

**Lagrange's equation:**

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = Q_\theta(t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[ma^2\dot{\theta}(t) - ma\dot{Z}(t)\sin\theta(t)\Big] + ma\dot{Z}(t)\dot{\theta}(t)\cos\theta(t) = -mga\sin\theta(t)$$

$$\Rightarrow \ddot{\theta}(t) - \frac{1}{a}\ddot{Z}(t)\sin\theta(t) = -\frac{g}{a}\sin\theta(t)$$

Substituting $Z(t) = Z_0\cos\Omega t$ gives $\ddot{Z}(t) = -\Omega^2 Z_0\cos(\Omega t)$:

$$\boxed{\ddot{\theta}(t) + \frac{\Omega^2 Z_0}{a}\cos(\Omega t)\sin\theta(t) + \frac{g}{a}\sin\theta(t) = 0}$$