

Worksheet 15

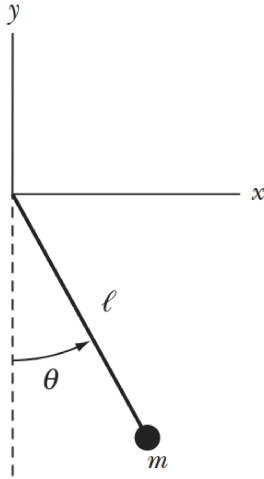
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Classical Mechanics II

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Problem 7.3



A point particle of mass m attached to a massless rod of length ℓ rotates about a frictionless pivot in a plane.

Holonomic constraints: particle is constrained to move in the xy -plane and length of the rod is fixed

$$\begin{cases} f_1(x, y, z, t) = z(t) = 0 \\ f_2(x, y, z, t) = x^2(t) + y^2(t) - \ell^2 = 0 \end{cases}$$

One degree of freedom; two possible generalized coordinates are: (1) $q_1 = x$; or (2) $q_1 = \theta$

Use d'Alembert's principle to obtain respective equations of motion for $x(t)$ and $\theta(t)$.

Solution

Method 1: Using x as the generalized coordinate

We have,

$$x^2(t) + y^2(t) - \ell^2 = 0 \Rightarrow \begin{cases} \delta y = \frac{x(t)}{\sqrt{\ell^2 - x^2(t)}} \delta x \\ \ddot{y}(t) = \frac{x^2(t)\ddot{x}(t) + \dot{x}^2(t)}{\sqrt{\ell^2 - x^2(t)}} + \frac{\dot{x}^2(t)x^2(t)}{[\ell^2 - x^2(t)]^{3/2}} \end{cases}$$

Position: $\mathbf{r}(t) = x(t)\hat{e}_x + y(t)\hat{e}_y$

Applied force: $\mathbf{F}^{(A)}(t) = -mg\hat{e}_y$

D'Alembert's principle:

$$\begin{aligned} & [\mathbf{F}^{(A)}(t) - m\ddot{\mathbf{r}}(t)] \cdot \delta \mathbf{r} = 0 \\ \Rightarrow & \ddot{x}(t)\delta x + \ddot{y}(t)\delta y + g\delta y = 0 \\ \Rightarrow & \ddot{x}(t) + [\ddot{y}(t) + g] \frac{x(t)}{\sqrt{\ell^2 - x^2(t)}} = 0 \end{aligned}$$

Finally,

$$\ddot{x}(t) = \frac{x(t)\dot{x}^2(t)}{\ell^2 - x^2(t)} - \frac{g}{\ell^2}x(t)\sqrt{\ell^2 - x^2(t)}$$

Method 2: Using θ as the generalized coordinate

From constraint:

$$\mathbf{r}(t) = \ell \sin \theta(t) \hat{e}_x - \ell \cos \theta(t) \hat{e}_y$$

$$\begin{cases} \dot{\mathbf{r}}(t) = \ell [\ddot{\theta}(t) \cos \theta(t) - \dot{\theta}^2(t) \sin \theta(t)] \hat{e}_x + \ell [\dot{\theta}(t) \sin \theta(t) + \dot{\theta}^2(t) \cos \theta(t)] \hat{e}_y \\ \delta \mathbf{r} = \frac{\partial \mathbf{r}}{\partial \theta} \delta \theta = \ell \delta \theta [\cos \theta(t) \hat{e}_x + \sin \theta(t) \hat{e}_y] \end{cases}$$

D'Alembert's principle:

$$\begin{aligned} & [\mathbf{F}^{(A)}(t) - m\ddot{\mathbf{r}}(t)] \cdot \delta \mathbf{r} = 0 \\ \Rightarrow & -mgl \sin \theta(t) \delta \theta - m\ell^2 [\ddot{\theta}(t) (\cos^2 \theta(t) + \sin^2 \theta(t))] \delta \theta = 0 \\ \Rightarrow & -mgl \sin \theta(t) - m\ell^2 \ddot{\theta}(t) = 0 \end{aligned}$$

Finally,

$$\ddot{\theta}(t) + \frac{g}{\ell} \sin \theta(t) = 0$$