Worksheet 14

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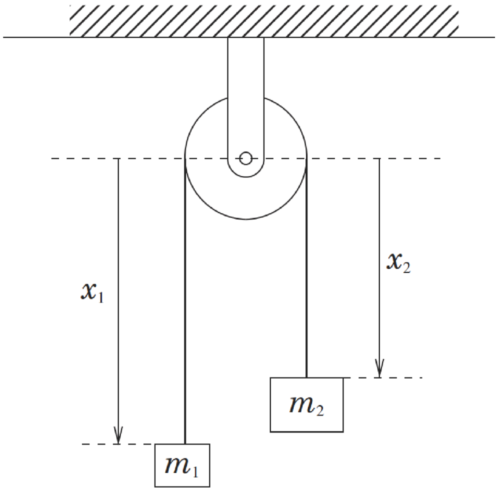
PC3261
Classical Mechanics II

November 21, 2025

Problem 6.4



Two masses m_1 and m_2 are suspended by an inextensible string which passes over a massless and frictionless pulley.

Holonomic constraint:

$$f(\boldsymbol{r}_1, \boldsymbol{r}_2) = x_1 + x_2 - \ell = 0$$

Applied forces:

$$\boldsymbol{F}_1^{(A)}(t) = m_1 g \hat{e}_1, \quad \boldsymbol{F}_2^{(A)}(t) = m_2 g \hat{e}_2$$

Use d'Alembert's principle to find the accelerations of the masses $\ddot{x}_1(t)$ and $\ddot{x}_2(t)$.

Solution

Constraint relations:

From the holonomic constraint:

$$f(\boldsymbol{r}_1, \boldsymbol{r}_2) = x_1 + x_2 - \ell = 0 \quad \Rightarrow \quad \delta x_1 = -\delta x_2 \quad \Leftrightarrow \quad \ddot{x}_1(t) = -\ddot{x}_2(t)$$

Apply d'Alembert's principle:

D'Alembert's principle states:

$$\sum_{\alpha} \left[\boldsymbol{F}_\alpha^{(A)}(t) - m_\alpha \ddot{\boldsymbol{r}}_{\alpha(t)} \right] \cdot \delta \boldsymbol{r}_\alpha = 0$$

Since the masses move only vertically, we have $\delta \boldsymbol{r}_1 = \delta x_1 \hat{e}_1$ and $\delta \boldsymbol{r}_2 = \delta x_2 \hat{e}_2$.

Expanding:

$$m_1 \ddot{\boldsymbol{r}}_1(t) \cdot \delta \boldsymbol{r}_1 + m_2 \ddot{\boldsymbol{r}}_2(t) \cdot \delta \boldsymbol{r}_2 = \boldsymbol{F}_1^{(A)}(t) \cdot \delta \boldsymbol{r}_1 + \boldsymbol{F}_2^{(A)}(t) \cdot \delta \boldsymbol{r}_2$$

$$m_1 \ddot{x}_1(t) \delta x_1 + m_2 \ddot{x}_2(t) \delta x_2 = m_1 g \delta x_1 + m_2 g \delta x_2$$

Using the constraint $\delta x_1 = -\delta x_2$ (or equivalently $\delta x_2 = -\delta x_1$):

$$m_1 \ddot{x}_1(t) \delta x_1 + [-m_2 \ddot{x}_1(t)](-\delta x_1) = m_1 g \delta x_1 + m_2 g(-\delta x_1)$$

$$(m_1 + m_2) \ddot{x}_1(t) \delta x_1 = (m_1 - m_2) g \delta x_1$$

Since δx_1 is arbitrary (non zero), we can divide both sides by δx_1:

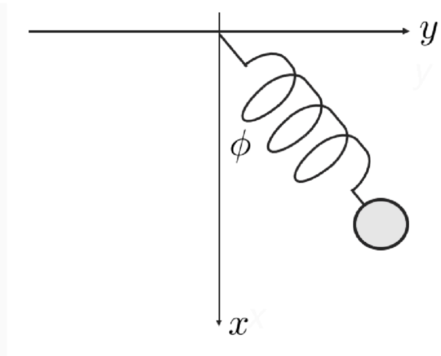
$$\ddot{x}_1(t) = \frac{m_1 - m_2}{m_1 + m_2} g$$

Using the constraint relation $\ddot{x}_2(t) = -\ddot{x}_1(t)$:

$$\ddot{x}_2(t) = -\frac{m_1 - m_2}{m_1 + m_2} g = \frac{m_2 - m_1}{m_1 + m_2} g$$

$$\boxed{\begin{cases} \ddot{x}_1(t) = \dfrac{m_1 - m_2}{m_1 + m_2} g \\ \ddot{x}_2(t) = \dfrac{m_2 - m_1}{m_1 + m_2} g \end{cases}}$$

Problem 6.5



A point particle of mass m attached to a massless spring of original length ℓ_0 and spring constant k rotates about a frictionless pivot in a plane.

Applied forces:

$$\begin{cases} \boldsymbol{F}_{\text{gravity}}(t) = mg\cos\varphi(t)\hat{e}_\rho - mg\sin\varphi(t)\hat{e}_\varphi \\ \boldsymbol{F}_{\text{spring}}(t) \;= -k[\rho(t) - \ell_0]\hat{e}_\rho \end{cases}$$

Use d'Alembert's principle to obtain equations of motion for $\rho(t)$ and $\varphi(t)$.

Solution

Express position, acceleration, and virtual displacement:

In polar coordinates, the position vector and its acceleration are:

$$\boldsymbol{r}(t) = \rho(t)\hat{e}_\rho$$

The acceleration in polar coordinates is:

$$\ddot{\boldsymbol{r}} = [\ddot{\rho}(t) - \rho(t)\dot{\varphi}^2(t)]\hat{e}_\rho + [\rho(t)\ddot{\varphi}(t) + 2\dot{\rho}(t)\dot{\varphi}(t)]\hat{e}_\varphi$$

The virtual displacement is:

$$\delta\boldsymbol{r} = \delta\rho\hat{e}_\rho + \rho(t)\delta\varphi\hat{e}_\varphi$$

Apply d'Alembert's principle:

D'Alembert's principle states:

$$\left[\boldsymbol{F}^{(A)}(t) - m\ddot{\boldsymbol{r}}(t)\right] \cdot \delta\boldsymbol{r} = 0$$

The total applied force is:

$$\boldsymbol{F}^{(A)}(t) = \boldsymbol{F}_{\text{gravity}}(t) + \boldsymbol{F}_{\text{spring}}(t) = [mg\cos\varphi(t) - k(\rho(t) - \ell_0)]\hat{e}_\rho - mg\sin\varphi(t)\hat{e}_\varphi$$

Substituting into d'Alembert's principle:

$$\begin{aligned} &[mg\cos\varphi(t) - k(\rho(t) - \ell_0)]\delta\rho - mg\sin\varphi(t)\rho(t)\delta\varphi \\ &-m[\ddot{\rho}(t) - \rho(t)\dot{\varphi}^2(t)]\delta\rho - m[\rho(t)\ddot{\varphi}(t) + 2\dot{\rho}(t)\dot{\varphi}(t)]\rho(t)\delta\varphi = 0 \end{aligned}$$

Collecting terms by $\delta\rho$ and $\delta\varphi$:

$$\begin{aligned} &\{mg\cos\varphi(t) - k[\rho(t) - \ell_0] - m[\ddot{\rho}(t) - \rho(t)\dot{\varphi}^2(t)]\}\delta\rho \\ &+\{-mg\sin\varphi(t) - m[\rho(t)\ddot{\varphi}(t) + 2\dot{\rho}(t)\dot{\varphi}(t)]\}\rho(t)\delta\varphi = 0 \end{aligned}$$

Since $\delta\rho$ and $\delta\varphi$ are independent and arbitrary, their coefficients must vanish:

Equations of motion:

$$\begin{cases} mg\cos\varphi(t) - k[\rho(t) - \ell_0] - m[\ddot{\rho}(t) - \rho(t)\dot{\varphi}^2(t)] = 0 \\ -mg\sin\varphi(t) - m[\rho(t)\ddot{\varphi}(t) + 2\dot{\rho}(t)\dot{\varphi}(t)] \qquad\qquad = 0 \end{cases}$$

Simplifying:

$$\boxed{\begin{cases} \ddot{\rho}(t) - \rho(t)\dot{\varphi}^2(t) \qquad = g\cos\varphi(t) - \frac{k}{m}[\rho(t) - \ell_0] \\ \rho(t)\ddot{\varphi}(t) + 2\dot{\rho}(t)\dot{\varphi}(t) = -g\sin\varphi(t) \end{cases}}$$

The second equation can also be written as:

$$\frac{\mathrm{d}}{\mathrm{d}t}[\rho^2(t)\dot{\varphi}(t)] = -\rho(t)g\sin\varphi(t)$$

which shows conservation of angular momentum in the absence of tangential forces.

Problem 6.6

A particle of mass m is suspended by a massless wire of length $r(t)$ to move on the surface of the sphere of radius $r(t)$.

Given: $r(t) = a + b\cos\omega t$, where $a > b > 0$

Holonomic constraint: $f(\boldsymbol{r}, t) = r(t) - a - b\cos\omega t = 0$

Applied force: $\boldsymbol{F}_{\text{gravity}}(t) = -mg\cos\theta(t)\hat{e}_r + mg\sin\theta(t)\hat{e}_\theta$

Use d'Alembert's principle to obtain equations of motion for $\theta(t)$ and $\varphi(t)$.

Solution

Kinematics of the constraint:

From the constraint $f(\boldsymbol{r}) = r(t) - a - b\cos\omega t = 0$, we have:

$$r(t) = a + b\cos\omega t$$

Taking time derivatives:

$$\dot{r}(t) = -b\omega\sin\omega t, \quad \ddot{r}(t) = -b\omega^2\cos\omega t$$

Acceleration in spherical coordinates:

The acceleration of a particle in spherical coordinates with time varying $r(t)$ is:

$$\begin{aligned} \boldsymbol{a}(t) = &\left[\ddot{r}(t) - r(t)\dot{\varphi}^2(t)\sin^2\theta(t) - r(t)\dot{\theta}^2(t)\right]\hat{e}_r \\ &+ \left[r(t)\ddot{\theta}(t) + 2\dot{r}(t)\dot{\theta}(t) - r(t)\dot{\varphi}^2(t)\sin\theta(t)\cos\theta(t)\right]\hat{e}_\theta \\ &+ \left[r(t)\ddot{\varphi}(t)\sin\theta(t) + 2\dot{r}(t)\dot{\varphi}(t)\sin\theta(t) + 2r(t)\dot{\theta}(t)\dot{\varphi}(t)\cos\theta(t)\right]\hat{e}_\varphi \end{aligned}$$

Virtual displacement:

Since the particle is constrained to the sphere of radius $r(t)$, the virtual displacement is:

$$\delta\boldsymbol{r} = r(t)\delta\theta\hat{e}_\theta + r(t)\sin\theta(t)\delta\varphi\hat{e}_\varphi$$

(No δr component since $r(t)$ is prescribed by the constraint.)

Apply d'Alembert's principle:

D'Alembert's principle:

$$\left[\boldsymbol{F}^{(A)}(t) - m\ddot{\boldsymbol{r}}(t)\right] \cdot \delta \boldsymbol{r} = 0$$

Computing the dot product:

$$\Big[mg\sin\theta(t) - m\Big(r(t)\ddot{\theta}(t) + 2\dot{r}(t)\dot{\theta}(t) - r(t)\dot{\varphi}^2(t)\sin\theta(t)\cos\theta(t)\Big)\Big]r(t)\delta\theta$$
$$+\Big[-m\Big(r(t)\ddot{\varphi}(t)\sin\theta(t) + 2\dot{r}(t)\dot{\varphi}(t)\sin\theta(t) + 2r(t)\dot{\theta}(t)\dot{\varphi}(t)\cos\theta(t)\Big)\Big]r(t)\sin\theta(t)\delta\varphi = 0$$

Since $\delta\theta$ and $\delta\varphi$ are independent and arbitrary:

$$\begin{cases} mgr(t)\sin\theta(t) - mr(t)\left[r(t)\ddot{\theta}(t) + 2\dot{r}(t)\dot{\theta}(t) - r(t)\dot{\varphi}^2(t)\sin\theta(t)\cos\theta(t)\right] & = 0 \\ -mr(t)\sin\theta(t)\left[r(t)\ddot{\varphi}(t)\sin\theta(t) + 2\dot{r}(t)\dot{\varphi}(t)\sin\theta(t) + 2r(t)\dot{\theta}(t)\dot{\varphi}(t)\cos\theta(t)\right] & = 0 \end{cases}$$

Simplify equations of motion:

Dividing and simplifying, and substituting $r(t) = a + b\cos\omega t$:

$\boldsymbol{\theta}$-equation:

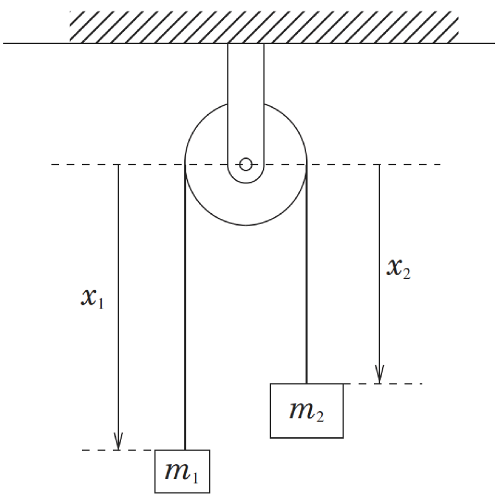
$$\boxed{(a + b\cos\omega t)\ddot{\theta}(t) - 2b\omega\dot{\theta}(t)\sin\omega t - (a + b\cos\omega t)\dot{\varphi}^2(t)\sin\theta(t)\cos\theta(t) = g\sin\theta(t)}$$

$\boldsymbol{\varphi}$-equation:

$$\boxed{(a + b\cos\omega t)\ddot{\varphi}(t)\sin\theta(t) - 2b\omega\dot{\varphi}(t)\sin\omega t\sin\theta(t) + 2(a + b\cos\omega t)\dot{\theta}(t)\dot{\varphi}(t)\cos\theta(t) = 0}$$

These are the coupled nonlinear equations of motion for a particle on a sphere with time varying radius.

Problem 7.1



Two masses m_1 and m_2 are suspended by an inextensible string which passes over a massless and frictionless pulley.

Holonomic constraint:

$$f(\boldsymbol{r}_1, \boldsymbol{r}_2, t) = x_1(t) + x_2(t) - \ell = 0$$

Applied forces:

$$\boldsymbol{F}_1^{(A)}(t) = m_1 g\hat{e}_1, \quad \boldsymbol{F}_2^{(A)}(t) = m_2 g\hat{e}_2$$

Use d'Alembert's principle with Lagrange multipliers to find the constrained forces.

Solution

Position vectors and kinematics:

The position vectors are:

$$\begin{cases} \boldsymbol{r}_1(t) = x_1(t)\hat{e}_1 \\ \boldsymbol{r}_2(t) = x_2(t)\hat{e}_2 \end{cases} \Rightarrow \begin{cases} \delta\boldsymbol{r}_1 = \delta x_1\hat{e}_1 \\ \delta\boldsymbol{r}_2 = \delta x_2\hat{e}_2 \end{cases} \Rightarrow \begin{cases} \ddot{\boldsymbol{r}}_1(t) = \ddot{x}_1(t)\hat{e}_1 \\ \ddot{\boldsymbol{r}}_2(t) = \ddot{x}_2(t)\hat{e}_2 \end{cases}$$

Constraint gradients:

From the constraint $f(\boldsymbol{r}_1, \boldsymbol{r}_2, t) = x_1(t) + x_2(t) - \ell = 0$:

$$\begin{cases} \dfrac{\partial f}{\partial \boldsymbol{r}_1} = \hat{e}_1 \\ \dfrac{\partial f}{\partial \boldsymbol{r}_2} = \hat{e}_2 \end{cases}$$

D'Alembert's principle with Lagrange multipliers:

The modified d'Alembert's principle with Lagrange multipliers is:

$$\sum_\alpha \left[\boldsymbol{F}_\alpha^{(A)}(t) + \lambda(t)\frac{\partial f}{\partial \boldsymbol{r}_\alpha} - m_\alpha \ddot{\boldsymbol{r}}_{\alpha(t)} \right] \cdot \delta\boldsymbol{r}_\alpha = 0$$

Substituting the known quantities:

$$[m_1 g + \lambda(t) - m_1\ddot{x}_1(t)]\delta x_1 + [m_2 g + \lambda(t) - m_2\ddot{x}_2(t)]\delta x_2 = 0$$

Since this must hold for all virtual displacements (not just those satisfying the constraint when using Lagrange multipliers), we get the system:

$$\begin{cases} m_1 g + \lambda(t) - m_1\ddot{x}_1(t) = 0 \\ m_2 g + \lambda(t) - m_2\ddot{x}_2(t) = 0 \\ x_1(t) + x_2(t) - \ell \qquad = 0 \end{cases}$$

Solve for accelerations:

From the constraint, taking two time derivatives:

$$x_1(t) + x_2(t) - \ell = 0 \quad \Rightarrow \quad \ddot{x}_1(t) + \ddot{x}_2(t) = 0$$

From the first two equations:

$$\begin{cases} m_1 g + \lambda(t) - m_1\ddot{x}_1(t) = 0 \\ m_2 g + \lambda(t) - m_2\ddot{x}_2(t) = 0 \end{cases} \Rightarrow \begin{cases} \ddot{x}_1(t) = \dfrac{m_1 - m_2}{m_1 + m_2} g \\ \ddot{x}_2(t) = \dfrac{m_2 - m_1}{m_1 + m_2} g \end{cases}$$

Find Lagrange multiplier and constraint forces:

Substituting $\ddot{x}_1(t)$ back into the first equation:

$$m_1 g + \lambda(t) - m_1\ddot{x}_1(t) = 0 \quad \Rightarrow \quad \lambda(t) = -\frac{2m_1 m_2}{m_1 + m_2} g$$

The constraint forces are given by:

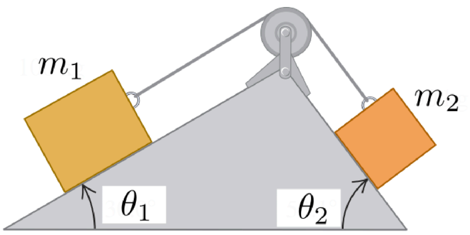
$$\boldsymbol{F}_i^{(C)}(t) = \lambda(t)\frac{\partial f}{\partial \boldsymbol{r}_i}$$

Therefore:

$$\begin{cases} \boldsymbol{F}_1^{(C)}(t) = \lambda(t)\frac{\partial f}{\partial \boldsymbol{r}_1} = -\dfrac{2m_1 m_2}{m_1+m_2} g\hat{e}_1 \\ \boldsymbol{F}_2^{(C)}(t) = \lambda(t)\frac{\partial f}{\partial \boldsymbol{r}_2} = -\dfrac{2m_1 m_2}{m_1+m_2} g\hat{e}_2 \end{cases}$$

These are the tension forces in the string acting on each mass. The negative sign indicates the forces act upward (opposite to the positive coordinate directions), as expected for tension.

Problem 7.2



Two masses m_1 and m_2 are located each on a smooth double inclined plane with angles θ_1 and θ_2 respectively. The masses are connected by a massless and inextensible string running over a massless and frictionless pulley.

Holonomic constraints:

$$\begin{cases} f_1(\boldsymbol{r}_1, \boldsymbol{r}_2, t) = x_1(t) + x_2(t) - \ell = 0 \\ f_2(\boldsymbol{r}_1, \boldsymbol{r}_2, t) = y_1(t) = 0 \\ f_3(\boldsymbol{r}_1, \boldsymbol{r}_2, t) = y_2(t) = 0 \end{cases}$$

Applied forces:

$$\boldsymbol{F}_1^{(A)}(t) = m_1 g \sin\theta_1 \hat{e}_{x_1} - m_1 g \cos\theta_1 \hat{e}_{y_1}$$

$$\boldsymbol{F}_2^{(A)}(t) = m_2 g \sin\theta_2 \hat{e}_{x_2} - m_2 g \cos\theta_2 \hat{e}_{y_2}$$

Use d'Alembert's principle with Lagrange multipliers to find the constrained forces.

Solution

Position vectors and virtual displacements:

The position vectors in the inclined plane coordinates are:

$$\begin{cases} \boldsymbol{r}_1(t) = x_1(t)\hat{e}_{x_1} + y_1(t)\hat{e}_{y_1} \\ \boldsymbol{r}_2(t) = x_2(t)\hat{e}_{x_2} + y_2(t)\hat{e}_{y_2} \end{cases} \Rightarrow \begin{cases} \delta\boldsymbol{r}_1 = \delta x_1 \hat{e}_{x_1} + \delta y_1 \hat{e}_{y_1} \\ \delta\boldsymbol{r}_2 = \delta x_2 \hat{e}_{x_2} + \delta y_2 \hat{e}_{y_2} \end{cases} \Rightarrow \begin{cases} \ddot{\boldsymbol{r}}_1(t) = \mathbb{0} \\ \ddot{\boldsymbol{r}}_2(t) = \mathbb{0} \end{cases}$$

(For equilibrium, accelerations are zero.)

Constraint gradients:

Computing the partial derivatives of the constraints:

$$\begin{cases} f_1(\boldsymbol{r}_1, \boldsymbol{r}_2, t) = x_1(t) + x_2(t) - \ell = 0 \\ f_2(\boldsymbol{r}_1, \boldsymbol{r}_2, t) = y_1(t) = 0 \\ f_3(\boldsymbol{r}_1, \boldsymbol{r}_2, t) = y_2(t) = 0 \end{cases}$$

The gradients are:

$$\begin{cases} \boldsymbol{F}_1^{(C)}(t) = \lambda_1(t)\frac{\partial f_1}{\partial \boldsymbol{r}_1} + \lambda_2(t)\frac{\partial f_2}{\partial \boldsymbol{r}_1} + \lambda_3(t)\frac{\partial f_3}{\partial \boldsymbol{r}_1} = \lambda_1(t)\hat{e}_{x_1} + \lambda_2(t)\hat{e}_{y_1} \\ \boldsymbol{F}_2^{(C)}(t) = \lambda_1(t)\frac{\partial f_1}{\partial \boldsymbol{r}_2} + \lambda_2(t)\frac{\partial f_2}{\partial \boldsymbol{r}_2} + \lambda_3(t)\frac{\partial f_3}{\partial \boldsymbol{r}_2} = \lambda_1(t)\hat{e}_{x_2} + \lambda_3(t)\hat{e}_{y_2} \end{cases}$$

Apply d'Alembert's principle with Lagrange multipliers:

For equilibrium with Lagrange multipliers:

$$\sum_\alpha \left[\boldsymbol{F}_\alpha^{(A)}(t) + \sum_i \lambda_i(t) \frac{\partial f_i}{\partial \boldsymbol{r}_\alpha} - m_\alpha \ddot{\boldsymbol{r}}_{\alpha(t)} \right] \cdot \delta \boldsymbol{r}_\alpha = 0$$

Since $\ddot{\boldsymbol{r}}_{\alpha(t)} = \mathbb{0}$ for equilibrium:

$$[m_1 g \sin\theta_1 + \lambda_1(t)]\delta x_1 + [-m_1 g\cos\theta_1 + \lambda_2(t)]\delta y_1$$
$$+[m_2 g \sin\theta_2 + \lambda_1(t)]\delta x_2 + [-m_2 g\cos\theta_2 + \lambda_3(t)]\delta y_2 = 0$$

Since all virtual displacements are independent in the Lagrange multiplier method:

$$\begin{cases} m_1 g \sin\theta_1 + \lambda_1(t) &= 0 \\ -m_1 g\cos\theta_1 + \lambda_2(t) &= 0 \\ m_2 g \sin\theta_2 + \lambda_1(t) &= 0 \\ -m_2 g\cos\theta_2 + \lambda_3(t) &= 0 \end{cases}$$

Solve for Lagrange multipliers:

From the equilibrium condition (from equations 1 and 3):

$$m_1 g \sin\theta_1 + \lambda_1(t) = 0 \quad \text{and} \quad m_2 g\sin\theta_2 + \lambda_1(t) = 0$$

This gives the equilibrium condition:

$$m_1 \sin\theta_1 = m_2 \sin\theta_2$$

The Lagrange multipliers are:

$$\begin{cases} \lambda_1(t) = -m_1 g\sin\theta_1 = -m_2 g \sin\theta_2 \\ \lambda_2(t) = m_1 g\cos\theta_1 \\ \lambda_3(t) = m_2 g\cos\theta_2 \end{cases}$$

Constraint forces:

$$\boxed{\begin{cases} \boldsymbol{F}_1^{(C)}(t) = \lambda_1(t)\hat{e}_{x_1} + \lambda_2(t)\hat{e}_{y_1} = -m_1 g\sin\theta_1 \hat{e}_{x_1} + m_1 g\cos\theta_1 \hat{e}_{y_1} \\ \boldsymbol{F}_2^{(C)}(t) = \lambda_1(t)\hat{e}_{x_2} + \lambda_3(t)\hat{e}_{y_2} = -m_2 g\sin\theta_2 \hat{e}_{x_2} + m_2 g\cos\theta_2 \hat{e}_{y_2} \end{cases}}$$

Interpretation:

- λ_1 represents the tension in the string (negative, acting to restrain motion along the incline)
- λ_2 and λ_3 represent the normal forces from the inclined planes on masses 1 and 2, respectively