Worksheet 13

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Classical Mechanics II

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Problem 6.1

A particle is confined to a moving surface where the equation of the surface is given by $f(\boldsymbol{r}, t) = 0$. Show that the virtual displacement is tangent to the surface at the same time.

Solution

Consider a particle constrained to the surface $f(\boldsymbol{r}, t) = 0$.

For a virtual displacement $\delta\boldsymbol{r}$ at fixed time, Taylor expansion gives:

$$f(\boldsymbol{r}(t) + \delta\boldsymbol{r}, t) = 0$$

Expanding to first order:

$$f(\boldsymbol{r}(t), t) + \nabla f(\boldsymbol{r}, t) \cdot \delta\boldsymbol{r} = 0$$

Since the particle is already on the surface, $f(\boldsymbol{r}(t), t) = 0$, therefore:

$$\nabla f(\boldsymbol{r}, t) \cdot \delta\boldsymbol{r} = 0$$

This shows that the virtual displacement $\delta\boldsymbol{r}$ is perpendicular to the gradient $\nabla f(\boldsymbol{r}, t)$, which is the normal to the surface. Hence, $\delta\boldsymbol{r}$ is tangent to the surface.

For an actual displacement, the particle moves to $\boldsymbol{r}(t + \mathrm{d}t)$ at time $t + \mathrm{d}t$:

$$f(\boldsymbol{r}(t) + \mathrm{d}\boldsymbol{r}, t + \mathrm{d}t) = 0$$

Expanding:

$$f(\boldsymbol{r}(t), t) + \nabla f(\boldsymbol{r}, t) \cdot \mathrm{d}\boldsymbol{r} + \frac{\partial f}{\partial t}\,\mathrm{d}t = 0$$

Again, since $f(\boldsymbol{r}(t), t) = 0$:

$$\boxed{\nabla f(\boldsymbol{r}, t) \cdot \mathrm{d}\boldsymbol{r} + \frac{\partial f}{\partial t}\,\mathrm{d}t = 0}$$

This equation shows that the actual displacement $\mathrm{d}\boldsymbol{r}$ has both a component tangent to the surface and a component accounting for the time evolution of the surface itself.

Problem 6.2

Show that the total virtual work by the constrained forces on the two particles connected by a rigid rod moving in the space is zero.

Solution

Consider two particles connected by a rigid rod of length ℓ. The constraint is:

$$f(\boldsymbol{r}_1, \boldsymbol{r}_2) = |\boldsymbol{r}_1 - \boldsymbol{r}_2|^2 - \ell^2 = 0$$

Taking the differential of the constraint:

$$\frac{\partial f}{\partial \boldsymbol{r}_1} \cdot \delta \boldsymbol{r}_1 + \frac{\partial f}{\partial \boldsymbol{r}_2} \cdot \delta \boldsymbol{r}_2 = 0$$

Computing the partial derivatives:

$$\frac{\partial f}{\partial \boldsymbol{r}_1} = 2(\boldsymbol{r}_1 - \boldsymbol{r}_2), \qquad \frac{\partial f}{\partial \boldsymbol{r}_2} = -2(\boldsymbol{r}_1 - \boldsymbol{r}_2)$$

Therefore:

$$(\boldsymbol{r}_1 - \boldsymbol{r}_2) \cdot (\delta \boldsymbol{r}_1 - \delta \boldsymbol{r}_2) = 0$$

The constraint forces on the two particles are equal and opposite, acting along the rod:

$$\boldsymbol{f}_1 = \lambda(\boldsymbol{r}_1 - \boldsymbol{r}_2) = -\boldsymbol{f}_2$$

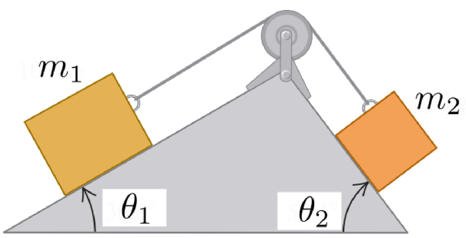
where λ is the Lagrange multiplier (tension/compression in the rod).

The total virtual work is:

$$\begin{aligned} \delta W &= \boldsymbol{f}_1 \cdot \delta \boldsymbol{r}_1 + \boldsymbol{f}_2 \cdot \delta \boldsymbol{r}_2 \\ &= \boldsymbol{f}_1 \cdot (\delta \boldsymbol{r}_1 - \delta \boldsymbol{r}_2) \\ &= \lambda(\boldsymbol{r}_1 - \boldsymbol{r}_2) \cdot (\delta \boldsymbol{r}_1 - \delta \boldsymbol{r}_2) \\ &= 0 \end{aligned}$$

Thus, the total virtual work done by the constraint forces is zero.

Problem 6.3



Two masses m_1 and m_2 are located each on a smooth double inclined plane with angles θ_1 and θ_2 respectively. The masses are connected by a massless and inextensible string running over a massless and frictionless pulley.

Holonomic constraints:

$$\begin{cases} f_1(\boldsymbol{r}_1, \boldsymbol{r}_2) = x_1 + x_2 - \ell = 0 \\ f_2(\boldsymbol{r}_1, \boldsymbol{r}_2) = y_1 = 0 \\ f_3(\boldsymbol{r}_1, \boldsymbol{r}_2) = y_2 = 0 \end{cases}$$

Applied forces:

$$\boldsymbol{F}_1^{(A)}(t) = m_1 g \sin\theta_1 \hat{e}_{x_1} - m_1 g \cos\theta_1 \hat{e}_{y_1}$$

$$\boldsymbol{F}_2^{(A)}(t) = m_2 g \sin\theta_2 \hat{e}_{x_2} - m_2 g \cos\theta_2 \hat{e}_{y_2}$$

Establish the condition for equilibrium from the principle of virtual work.

Solution

Determine virtual displacements from constraints:

Given the holonomic constraints:

$$\begin{cases} f_1(\boldsymbol{r}_1, \boldsymbol{r}_2) = x_1 + x_2 - \ell = 0 \\ f_2(\boldsymbol{r}_1, \boldsymbol{r}_2) = y_1 = 0 \\ f_3(\boldsymbol{r}_1, \boldsymbol{r}_2) = y_2 = 0 \end{cases}$$

Taking virtual variations:

$$\begin{cases} \delta x_1 = -\delta x_2 \\ \delta y_1 = 0 \\ \delta y_2 = 0 \end{cases}$$

Therefore, the virtual displacements are:

$$\begin{cases} \delta \boldsymbol{r}_1 = \delta x_1 \hat{e}_{x_1} \\ \delta \boldsymbol{r}_2 = \delta x_2 \hat{e}_{x_2} \end{cases}$$

Apply principle of virtual work:

The principle of virtual work states that for equilibrium:

$$\sum_\alpha \boldsymbol{F}_\alpha^{(A)}(t) \cdot \delta \boldsymbol{r}_\alpha = 0$$

Computing the virtual work:

$$\begin{aligned} \boldsymbol{F}_1^{(A)} \cdot \delta \boldsymbol{r}_1 + \boldsymbol{F}_2^{(A)} \cdot \delta \boldsymbol{r}_2 &= \left(m_1 g \sin\theta_1 \hat{e}_{x_1} - m_1 g \cos\theta_1 \hat{e}_{y_1}\right) \cdot \left(\delta x_1 \hat{e}_{x_1}\right) \\ &\quad + \left(m_2 g \sin\theta_2 \hat{e}_{x_2} - m_2 g \cos\theta_2 \hat{e}_{y_2}\right) \cdot \left(\delta x_2 \hat{e}_{x_2}\right) \\ &= m_1 g \delta x_1 \sin\theta_1 + m_2 g \delta x_2 \sin\theta_2 \end{aligned}$$

Using the constraint $\delta x_1 = -\delta x_2$:

$$\Rightarrow m_1 g \delta x_1 \sin\theta_1 + m_2 g \delta x_2 \sin\theta_2 = 0$$

$$\Rightarrow (m_1 g \sin\theta_1 - m_2 g \sin\theta_2) \delta x_1 = 0$$

Since δx_1 is arbitrary (non-zero in general), we obtain the equilibrium condition:

$$\boxed{m_1 \sin\theta_1 = m_2 \sin\theta_2}$$

This is the condition for equilibrium of the two-mass system on the double inclined plane.