Worksheet 12

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Classical Mechanics II

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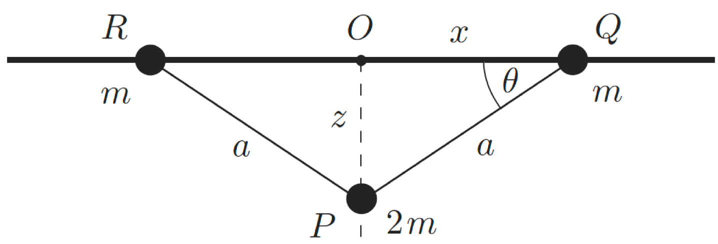
Problem 5.7

A ball P of mass $2m$ is suspended by two light inextensible strings of length a from two sliders Q and R, each of mass m, which can move on a smooth horizontal rail. The system moves symmetrically so that O, the midpoint of Q and R, remains fixed and P moves on the downward vertical through O. Initially, the system is released from rest with the three particles in a straight line and with the strings taut. Ignore gravitational forces between masses.



Tension forces exerted by the inextensible strings do zero work in total. Why?

Derive the first order differential equation governing the dynamics of the system.

Solution

Proof that tension forces do zero work:

Given that $\left[\boldsymbol{r}_\alpha(t) - \boldsymbol{r}_\beta(t)\right] \cdot \left[\boldsymbol{r}_\alpha(t) - \boldsymbol{r}_\beta(t)\right] = \text{constant}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\boldsymbol{r}_\alpha(t) - \boldsymbol{r}_\beta(t)\right] \cdot \left[\boldsymbol{r}_\alpha(t) - \boldsymbol{r}_\beta(t)\right] = 0$$

$$\Rightarrow \left[\dot{\boldsymbol{r}}_{\alpha(t)} - \dot{\boldsymbol{r}}_{\beta(t)}\right] \cdot \left[\boldsymbol{r}_\alpha(t) - \boldsymbol{r}_\beta(t)\right] + \left[\boldsymbol{r}_\alpha(t) - \boldsymbol{r}_\beta(t)\right] \cdot \left[\dot{\boldsymbol{r}}_{\alpha(t)} - \dot{\boldsymbol{r}}_{\beta(t)}\right] = 0$$

$$\Rightarrow 2\left[\boldsymbol{r}_\alpha(t) - \boldsymbol{r}_\beta(t)\right] \cdot \left[\dot{\boldsymbol{r}}_{\alpha(t)} - \dot{\boldsymbol{r}}_{\beta(t)}\right] = 0$$

$$\Rightarrow \left[\boldsymbol{r}_\alpha(t) - \boldsymbol{r}_\beta(t)\right] \cdot \left[\dot{\boldsymbol{r}}_\alpha - \dot{\boldsymbol{r}}_\beta\right) = 0$$

The string tension $\boldsymbol{f}_{\alpha\beta}$ is parallel to $\left(\boldsymbol{r}_\alpha - \boldsymbol{r}_\beta\right)$, so $\boldsymbol{f}_{\alpha\beta} \cdot \left(\dot{\boldsymbol{r}}_\alpha - \dot{\boldsymbol{r}}_\beta\right) = 0$.

Therefore,

$$W = \int_{t_1}^{t_2} \boldsymbol{f}_{\alpha\beta}(t) \cdot \dot{\boldsymbol{r}}_{\alpha(t)}\,\mathrm{d}(t) + \int_{t_1}^{t_2} \boldsymbol{f}_{\beta\alpha}(t) \cdot \dot{\boldsymbol{r}}_{\beta(t)}\,\mathrm{d}(t) = \int_{t_1}^{t_2} \boldsymbol{f}_{\alpha\beta}(t) \cdot \left[\dot{\boldsymbol{r}}_{\alpha(t)} - \dot{\boldsymbol{r}}_{\beta(t)}\right] \mathrm{d}(t) = 0$$

This shows that tension forces exerted by inextensible strings do zero work in total.

Dynamics of the system:

For the three-particle system, we set up coordinates with O at origin. Due to symmetric motion, Q and R (mass m each) move equal distances from O along y-axis, while P (mass $2m$) moves vertically below O. With string length a and angle θ from vertical:

$$\begin{cases} \boldsymbol{r}_1(t) = \boldsymbol{r}_R(t) = -a\cos\theta(t)\hat{e}_y \\ \boldsymbol{r}_2(t) = \boldsymbol{r}_Q(t) = +a\cos\theta(t)\hat{e}_y \\ \boldsymbol{r}_3(t) = \boldsymbol{r}_P(t) = -a\sin\theta(t)\hat{e}_z \end{cases} \Rightarrow \begin{cases} \dot{\boldsymbol{r}}_1(t) = +a\dot{\theta}(t)\sin\theta(t)\hat{e}_y \\ \dot{\boldsymbol{r}}_2(t) = -a\dot{\theta}(t)\sin\theta(t)\hat{e}_y \\ \dot{\boldsymbol{r}}_3(t) = -a\dot{\theta}(t)\cos\theta(t)\hat{e}_z \end{cases}$$

The kinetic energy is:

$$\begin{aligned} T(t) &= \frac{1}{2}ma^2\dot{\theta}^2\sin^2\theta + \frac{1}{2}ma^2\dot{\theta}^2\sin^2\theta + \frac{1}{2}(2m)a^2\dot{\theta}^2\cos^2\theta \\ &= ma^2\dot{\theta}^2(\sin^2\theta + \cos^2\theta) = ma^2\dot{\theta}^2(t) \end{aligned}$$

The potential energy due to gravity (taking rail level as reference):

$$U^{\text{ext}}(\boldsymbol{r}_1(t), \boldsymbol{r}_2(t), \boldsymbol{r}_3(t)) = mg \cdot 0 + mg \cdot 0 + 2mg(-a\sin\theta(t)) = -2mga\sin\theta(t)$$

The external non-conservative forces (rail normal forces) do zero work: $W_{\text{nc}} = 0$

For internal constraint forces (string tensions), since the strings are inextensible: $W_{\text{strings}} = 0$

Since all constraint forces do zero work: $U^{\text{int}}(\boldsymbol{r}_1(t), \boldsymbol{r}_2(t), \boldsymbol{r}_3(t)) = 0$

The total energy is:

$$E(t) = -2mga\sin\theta(t) + ma^2\dot{\theta}^2(t)$$

Initially (particles in straight line, released from rest): $\theta(0) = 0$, $\dot{\theta}(0) = 0$

$$E(0) = ma^2 \cdot 0^2 - 2mga\sin(0) = 0$$

By conservation of energy,

$$E(t) = E(0) = 0 \quad \Rightarrow \quad ma^2\dot{\theta}^2(t) - 2mga\sin\theta(t) = 0$$

$$\boxed{\dot{\theta}^2(t) - \frac{2g}{a}\sin\theta(t) = 0}$$

Problem 5.8

A block of mass m is projected from $x(0) = +a$ with $\dot{x}(0) = -u$ along the x-axis under action of a spring with spring constant k. Obtain the first order differential equation governing the dynamics of the block and solve for $x(t)$.

Solution

The total energy of the system is:

$$E(t) = U(x(t)) + \frac{1}{2}m\dot{x}^2(t) = \frac{1}{2}kx^2(t) + \frac{1}{2}m\dot{x}^2(t)$$

At $t = 0$:

$$E \equiv E(0) = \frac{1}{2}ka^2 + \frac{1}{2}mu^2 > 0$$

By conservation of energy:

$$
\begin{aligned}
&\frac{1}{2}kx^2(t)+\frac{1}{2}m\dot{x}^2(t)=E\\
&\Rightarrow \dot{x}(t)=\pm\sqrt{\frac{2}{m}}\sqrt{E-\frac{1}{2}kx^2(t)}\\
&\Rightarrow \dot{x}(t)=-\sqrt{\frac{2}{m}}\sqrt{E-\frac{1}{2}kx^2(t)} \qquad [\because \dot{x}(0)=-u<0]\\
&\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}\sqrt{E-\frac{1}{2}kx^2}}=-\sqrt{\frac{2}{m}}\,\mathrm{d}(t)\\
&\Rightarrow \int_a^x \frac{\mathrm{d}(x')}{\sqrt{E-\frac{1}{2}k(x')^2}}=-\sqrt{\frac{2}{m}}t\\
&\Rightarrow t=-\sqrt{\frac{m}{k}}\left[\arcsin\left(\sqrt{\frac{k}{2E}}x\right)-\arcsin\left(\sqrt{\frac{k}{2E}}a\right)\right]\\
&\Rightarrow -\sqrt{\frac{k}{m}}t=\arcsin\left(\sqrt{\frac{k}{2E}}x\right)-\arcsin\left(\sqrt{\frac{k}{2E}}a\right)\\
&\Rightarrow \arcsin\left(\sqrt{\frac{k}{2E}}x\right)=-\sqrt{\frac{k}{m}}t+\arcsin\left(\sqrt{\frac{k}{2E}}a\right)\\
&\Rightarrow \sqrt{\frac{k}{2E}}x=\sin\left[-\sqrt{\frac{k}{m}}t+\arcsin\left(\sqrt{\frac{k}{2E}}a\right)\right]\\
&\Rightarrow x(t)=\sqrt{\frac{2E}{k}}\sin\left[-\sqrt{\frac{k}{m}}t+\arcsin\left(\sqrt{\frac{k}{2E}}a\right)\right]
\end{aligned}
$$

Substituting $E=\frac{1}{2}ka^2+\frac{1}{2}mu^2$ and using trigonometric identities:

$$
\begin{aligned}
x(t)&=\sqrt{a^2+\frac{mu^2}{k}}\sin\left[-\sqrt{\frac{k}{m}}t+\arcsin\left(\frac{a}{\sqrt{a^2+\frac{mu^2}{k}}}\right)\right]\\
\Rightarrow x(t)&=\sqrt{a^2+\frac{mu^2}{k}}\left[\sin\left(\arcsin\left(\frac{a}{\sqrt{a^2+\frac{mu^2}{k}}}\right)\right)\cos\left(\sqrt{\frac{k}{m}}t\right)\right.\\
&\qquad\left.-\cos\left(\arcsin\left(\frac{a}{\sqrt{a^2+\frac{mu^2}{k}}}\right)\right)\sin\left(\sqrt{\frac{k}{m}}t\right)\right]\\
\Rightarrow x(t)&=\sqrt{a^2+\frac{mu^2}{k}}\left[\frac{a}{\sqrt{a^2+\frac{mu^2}{k}}}\cos\left(\sqrt{\frac{k}{m}}t\right)-\frac{\sqrt{\frac{mu^2}{k}}}{\sqrt{a^2+\frac{mu^2}{k}}}\sin\left(\sqrt{\frac{k}{m}}t\right)\right]
\end{aligned}
$$

$$
\boxed{x(t)=a\cos\left(\sqrt{\frac{k}{m}}t\right)-\sqrt{\frac{m}{k}}u\sin\left(\sqrt{\frac{k}{m}}t\right)}
$$

Problem 5.9

Derive the expression for the approximate period for small amplitude oscillations about $x = x_0$.

Solution

For small oscillations about an equilibrium point x_0, we expand the potential in a Taylor series:

$$U(x) \approx U(x_0) + \frac{1}{2}U''(x_0)(x - x_0)^2$$

where $U'(x_0) = 0$ at equilibrium. Let A be the amplitude of oscillation, so the turning points are $x_1 = x_0 - A$ and $x_2 = x_0 + A$.

The energy at the turning points is:

$$E = U(x_1) = U(x_0) + \frac{1}{2}U''(x_0)(x_1 - x_0)^2 = U(x_0) + \frac{1}{2}U''(x_0)A^2$$

$$E = U(x_2) = U(x_0) + \frac{1}{2}U''(x_0)(x_2 - x_0)^2 = U(x_0) + \frac{1}{2}U''(x_0)A^2$$

Therefore:

$$E - U(x) = U(x_0) + \frac{1}{2}U''(x_0)A^2 - \left[U(x_0) + \frac{1}{2}U''(x_0)(x - x_0)^2\right]$$

$$\Rightarrow E - U(x) = \frac{1}{2}U''(x_0)\left[A^2 - (x - x_0)^2\right]$$

The period is given by:

$$T = 4\sqrt{\frac{m}{2}}\int_{x_0}^{x_0+A}\frac{\mathrm{d}(x)}{\sqrt{E - U(x)}}$$

$$\Rightarrow T = 4\sqrt{\frac{m}{2}}\int_{x_0}^{x_0+A}\frac{\mathrm{d}(x)}{\sqrt{\frac{1}{2}U''(x_0)\left[A^2 - (x - x_0)^2\right]}}$$

$$\Rightarrow T = 4\sqrt{\frac{m}{U''(x_0)}}\int_{x_0}^{x_0+A}\frac{\mathrm{d}(x)}{\sqrt{A^2 - (x - x_0)^2}}$$

Making the substitution $x - x_0 = Au$:

$$\Rightarrow T = 4\sqrt{\frac{m}{U''(x_0)}}\int_0^1\frac{A\,\mathrm{d}(u)}{\sqrt{A^2 - A^2u^2}}$$

$$\Rightarrow T = 4\sqrt{\frac{m}{U''(x_0)}}\int_0^1\frac{\mathrm{d}(u)}{\sqrt{1 - u^2}}$$

The integral evaluates to:

$$\int_0^1\frac{\mathrm{d}(u)}{\sqrt{1 - u^2}} = [\arcsin(u)]_0^1 = \arcsin(1) - \arcsin(0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

Therefore:

$$\boxed{T = 2\pi\sqrt{\frac{m}{U''(x_0)}}}$$

Problem 5.10

Obtain a second order differential equation for $x(t)$ for the inverted harmonic oscillator with $E > 0$ and initial conditions $x(0)$ and $\dot{x}(0)$. Solve for $x(t)$.

Solution

For the inverted harmonic oscillator, the potential is:

$$U(x) = -\frac{1}{2}m\omega_0^2 x^2$$

The total energy is:

$$E(t) = \frac{1}{2}m\dot{x}^2(t) - \frac{1}{2}m\omega_0^2 x^2(t)$$

At $t = 0$:

$$E = \frac{1}{2}m\dot{x}^2(0) - \frac{1}{2}m\omega_0^2 x^2(0)$$

To find the second-order differential equation, we differentiate the energy equation:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = m\dot{x}(t)\ddot{x}(t) - m\omega_0^2 x(t)\dot{x}(t) = 0$$

$$\Rightarrow m\dot{x}(t)\left[\ddot{x}(t) - \omega_0^2 x(t)\right] = 0$$

Since energy is conserved and $\dot{x}(t) \neq 0$ (for $E > 0$):

$$\ddot{x}(t) - \omega_0^2 x(t) = 0$$

This is the second-order differential equation. The general solution is:

$$x(t) = C_1 \cosh(\omega_0 t) + C_2 \sinh(\omega_0 t)$$

Applying initial conditions:

$$\begin{cases} x(0) = C_1 \cosh(0) + C_2 \sinh(0) = C_1 \\ \dot{x}(0) = \omega_0 C_1 \sinh(0) + \omega_0 C_2 \cosh(0) = \omega_0 C_2 \end{cases} \Rightarrow \begin{cases} C_1 = x(0) \\ C_2 = \frac{\dot{x}(0)}{\omega_0} \end{cases}$$

Therefore:

$$\boxed{x(t) = x(0)\cosh(\omega_0 t) + \frac{\dot{x}(0)}{\omega_0}\sinh(\omega_0 t)}$$

Alternative Method: From conservation of energy:

$$\frac{1}{2}m\dot{x}^2(t) - \frac{1}{2}m\omega_0^2 x^2(t) = E$$

$$\Rightarrow \dot{x}^2(t) = \frac{2E}{m} + \omega_0^2 x^2(t)$$

$$\Rightarrow \dot{x}(t) = \pm\sqrt{\frac{2E}{m} + \omega_0^2 x^2(t)}$$

For the inverted harmonic oscillator with $E > 0$, the motion is unbounded. We choose the positive sign for $t > 0$:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sqrt{\frac{2E}{m} + \omega_0^2 x^2}$$

$$\Rightarrow \frac{\mathrm{d}x}{\sqrt{\frac{2E}{m}+\omega_0^2x^2}} = \mathrm{d}t$$

$$\Rightarrow t = \int_{x(0)}^{x(t)} \frac{\mathrm{d}x'}{\sqrt{\frac{2E}{m}+\omega_0^2(x')^2}} \qquad \left[\text{Let } \alpha = \sqrt{\frac{2E}{m\omega_0^2}}\right]$$

$$\Rightarrow t = \int_{x(0)}^{x(t)} \frac{\mathrm{d}x'}{\sqrt{\omega_0^2\left(\alpha^2+(x')^2\right)}}$$

$$\Rightarrow t = \frac{1}{\omega_0}\int_{x(0)}^{x(t)} \frac{\mathrm{d}x'}{\sqrt{\alpha^2+(x')^2}}$$

$$\Rightarrow t = \frac{1}{\omega_0}\left[\sinh^{-1}\left(\frac{x'}{\alpha}\right)\right]_{x(0)}^{x(t)} \qquad [\text{using standard integral}]$$

$$\Rightarrow t = \frac{1}{\omega_0}\left[\sinh^{-1}\left(\frac{x(t)}{\alpha}\right) - \sinh^{-1}\left(\frac{x(0)}{\alpha}\right)\right]$$

$$\Rightarrow \omega_0 t = \sinh^{-1}\left(\frac{x(t)}{\alpha}\right) - \sinh^{-1}\left(\frac{x(0)}{\alpha}\right)$$

$$\Rightarrow \sinh^{-1}\left(\frac{x(t)}{\alpha}\right) = \omega_0 t + \sinh^{-1}\left(\frac{x(0)}{\alpha}\right)$$

$$\Rightarrow \frac{x(t)}{\alpha} = \sinh\left[\omega_0 t + \sinh^{-1}\left(\frac{x(0)}{\alpha}\right)\right]$$

$$\Rightarrow x(t) = \alpha\sinh\left[\omega_0 t + \sinh^{-1}\left(\frac{x(0)}{\alpha}\right)\right]$$

$$\Rightarrow x(t) = \alpha\left[\sinh(\omega_0 t)\cosh\left(\sinh^{-1}\left(\frac{x(0)}{\alpha}\right)\right) + \cosh(\omega_0 t)\sinh\left(\sinh^{-1}\left(\frac{x(0)}{\alpha}\right)\right)\right]$$

$$\Rightarrow x(t) = \alpha\left[\sinh(\omega_0 t)\sqrt{1+\left(\frac{x(0)}{\alpha}\right)^2} + \cosh(\omega_0 t)\frac{x(0)}{\alpha}\right] \qquad [\text{using} \sinh^{-1} \text{identities}]$$

$$\Rightarrow x(t) = x(0)\cosh(\omega_0 t) + \frac{1}{\omega_0}\sqrt{\frac{2E}{m}+x(0)^2}\sinh(\omega_0 t)$$

$$\boxed{x(t) = x(0)\cosh(\omega_0 t) + \frac{\dot{x}(0)}{\omega_0}\sinh(\omega_0 t)}$$