Worksheet 11

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Classical Mechanics II

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Problem 5.4

Show that the total mechanical energy with time-independent potential energy is a constant of motion.

Solution

Given relationships:

$$E(t) \equiv U(\boldsymbol{r}(t)) + T(t), \qquad \boldsymbol{F}(\boldsymbol{r}) = -\boldsymbol{\nabla} U(\boldsymbol{r}), \qquad \boldsymbol{F}(\boldsymbol{r}(t)) = m\ddot{\boldsymbol{r}}(t)$$

Taking the time derivative of total energy:

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[U(\boldsymbol{r}(t)) + \frac{m}{2}\dot{\boldsymbol{r}}(t)\cdot\dot{\boldsymbol{r}}(t)\right] \\ &= \boldsymbol{\nabla} U(\boldsymbol{r}(t))\cdot\dot{\boldsymbol{r}}(t) + m\ddot{\boldsymbol{r}}(t)\cdot\dot{\boldsymbol{r}}(t) \\ &= -\boldsymbol{F}(\boldsymbol{r}(t))\cdot\dot{\boldsymbol{r}}(t) + \boldsymbol{F}(\boldsymbol{r}(t))\cdot\dot{\boldsymbol{r}}(t) \\ &= 0\end{aligned}$$

Therefore, the total mechanical energy is conserved when the potential energy is time-independent.

Problem 5.5

The electrostatic force on a point charge q located at $\boldsymbol{r}$ due to a fixed point charge Q at the origin is given by

$$\boldsymbol{F}(\boldsymbol{r}) = \frac{Qq}{4\pi\varepsilon_0 r^2}\hat{e}_r$$

Show that it is conservative and find the corresponding potential energy.

Solution

Given force:

$$\boldsymbol{F}(\boldsymbol{r}) = \frac{Qq}{4\pi\varepsilon_0 r^2}\hat{e}_r \equiv f(r)\hat{e}_r, \qquad \text{where} \quad f(r) = \frac{Qq}{4\pi\varepsilon_0 r^2}$$

Testing for conservativity using curl:

$$\boldsymbol{\nabla}\times\boldsymbol{F}(\boldsymbol{r}) = \frac{1}{h_1h_2h_3}\det\left(\begin{pmatrix} h_1\hat{e}_r & h_2\hat{e}_\theta & h_3\hat{e}_\varphi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ h_1F_r & h_2F_\theta & h_3F_\varphi\end{pmatrix}\right) = \frac{1}{r^2\sin\theta}\det\left(\begin{pmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\hat{e}_\varphi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ f(r) & 0 & 0\end{pmatrix}\right) = 0$$

Computing the gradient of potential energy:

$$
\begin{aligned}
\nabla U(\boldsymbol{r}) &= \frac{1}{h_1}\frac{\partial U(r)}{\partial r}\hat{e}_r + \frac{1}{h_2}\frac{\partial U(r)}{\partial \theta}\hat{e}_\theta + \frac{1}{h_3}\frac{\partial U(r)}{\partial \varphi}\hat{e}_\varphi \\
&= \frac{\partial U(r)}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial U(r)}{\partial \theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial U(r)}{\partial \varphi}\hat{e}_\varphi \\
&= \frac{\mathrm{d}}{\mathrm{d}r}\left[-\int_{r_0}^{r} f(r')dr'\right]\hat{e}_r = -f(r)\hat{e}_r
\end{aligned}
$$

Therefore:

$$
\boldsymbol{F}(\boldsymbol{r}) = -\nabla U(\boldsymbol{r}) = f(r)\hat{e}_r
$$

Computing the potential energy:

$$
\begin{aligned}
U(r) &= -\int_{r_0}^{r} f(r')dr' = -\frac{Qq}{4\pi\varepsilon_0}\int_{r_0}^{r}\frac{1}{r'^2}dr' \\
&= -\frac{Qq}{4\pi\varepsilon_0}\left[-\frac{1}{r'}\right]_{r_0}^{r} = \frac{Qq}{4\pi\varepsilon_0}\left(\frac{1}{r} - \frac{1}{r_0}\right)
\end{aligned}
$$

The electrostatic force is conservative, and the corresponding potential energy is

$$
U(r) = \frac{Qq}{4\pi\varepsilon_0}\left(\frac{1}{r} - \frac{1}{r_0}\right)
$$

.

Problem 5.6

Show that the total mechanical energy with time-dependent potential energy is not a constant of motion.

Solution

We have,

$$
\begin{cases}
T(t) = \frac{1}{2}m\dot{\boldsymbol{r}}(t)\cdot\dot{\boldsymbol{r}}(t) \\
U(\boldsymbol{r}, t) = -\int_{r_0}^{\boldsymbol{r}} \boldsymbol{F}(\boldsymbol{r}', t)\cdot d\boldsymbol{r}'
\end{cases}
$$

$$
\Longrightarrow \begin{cases}
dT = m\ddot{\boldsymbol{r}}(t)\cdot\dot{\boldsymbol{r}}(t)dt = \boldsymbol{F}(\boldsymbol{r}(t), t)\cdot\dot{\boldsymbol{r}}(t)dt = -\nabla U(\boldsymbol{r}(t), t)\cdot d\boldsymbol{r} \\
dU = \nabla U(\boldsymbol{r}(t), t)\cdot d\boldsymbol{r} + \dfrac{\partial U(\boldsymbol{r}(t), t)}{\partial t}dt
\end{cases}
$$

So,

$$
dE = dT + dU = \frac{\partial U(\boldsymbol{r}(t), t)}{\partial t}dt \Longrightarrow \frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\partial U(\boldsymbol{r}(t), t)}{\partial t} \neq 0
$$

Therefore, when the potential energy has explicit time dependence, the total mechanical energy is **not** conserved.