Worksheet 10

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Classical Mechanics II

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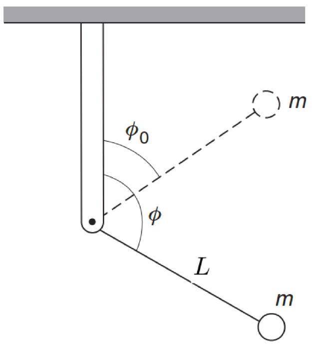
Problem 5.1



A pendulum consists of a light rigid rod of length L pivoted at one end with mass m attached at the other end. The pendulum is released from rest at angle φ_0.

Obtain the speed of the mass m when the rod is at an angle φ from work-energy theorem.

Solution

Position vector of the mass:

$$\boldsymbol{r}(t) = L\sin\varphi(t)\hat{e}_y + L\cos\varphi(t)\hat{e}_z \implies d\boldsymbol{r} = L\cos\varphi(t)d\varphi\hat{e}_y - L\sin\varphi(t)d\varphi\hat{e}_z$$

Forces acting on the mass:

$$\boldsymbol{F}(t) = \boldsymbol{W}(t) + \boldsymbol{N}(t) = -mg\hat{e}_z - \left[N(t)\sin\varphi(t)\hat{e}_y + N(t)\cos\varphi(t)\hat{e}_z\right]$$

Work done by all forces:

$$\begin{aligned} W(\boldsymbol{r}(0)\to\boldsymbol{r}(t)) &= \int_{\boldsymbol{r}(0)}^{\boldsymbol{r}(t)} \boldsymbol{F}(t)\cdot d\boldsymbol{r} = \int_{\boldsymbol{r}(0)}^{\boldsymbol{r}(t)} \boldsymbol{W}(t)\cdot d\boldsymbol{r} + \int_{\boldsymbol{r}(0)}^{\boldsymbol{r}(t)} \boldsymbol{N}(t)\cdot d\boldsymbol{r} \\ &= \int_{\varphi(0)}^{\varphi(t)} -mgL\sin\varphi(t)d\varphi = mgL[\cos\varphi(0) - \cos\varphi(t)] \end{aligned}$$

Note

The constraint force $\boldsymbol{N}(t)$ is always perpendicular to the displacement, so:

$$\int_{\boldsymbol{r}(0)}^{\boldsymbol{r}(t)} \boldsymbol{N}(t)\cdot d\boldsymbol{r} = 0$$

Applying work-energy theorem:

$$T(t) - T(0) = W(\boldsymbol{r}(0)\to\boldsymbol{r}(t)) \implies \frac{m}{2}v^2(t) - \frac{m}{2}v^2(0) = mgL[\cos\varphi(0) - \cos\varphi(t)]$$

$$\implies v(t) = \sqrt{2gL[\cos\varphi(0) - \cos\varphi(t)] + v^2(0)}$$

$$\boxed{v(t) = \sqrt{2gL[\cos\varphi_0 - \cos\varphi(t)]}}$$

Problem 5.2

Gravitational force acting on a mass m at a distance r from the center of Earth of mass M:

$$\boldsymbol{F}(\boldsymbol{r}) = -\frac{GMm}{r^2}\hat{e}_r$$

Mass m is projected from the surface of the Earth $r = R_e$ with an initial speed v_0 at an angle α from the vertical.

Escape speed for the mass m to escape Earth's gravitational field is independent of the launching direction:

$$v_{\text{escape}} = \sqrt{2gR_e}$$

Obtain the expression for the escape speed from work-energy theorem. Assume gravitational force is the only force and ignore the rotation of the Earth.

Solution

Gravitational force and displacement:

$$\boldsymbol{F}(\boldsymbol{r}) = -\frac{GMm}{r^2}\hat{e}_r$$

$$d\boldsymbol{r} = dr\hat{e}_r + rd\theta\hat{e}_\theta + r\sin\theta d\varphi\hat{e}_\varphi$$

Work done by gravitational force:

$$\begin{aligned}
W(\boldsymbol{r}_1 \to \boldsymbol{r}_2) &= \int_{r_1}^{r_2} \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} = -\int_{r_1}^{r_2} \frac{GMm}{r^2} dr \\
&= -GMm\int_{r_1}^{r_2} \frac{1}{r^2} dr = -GMm\left[-\frac{1}{r}\right]_{r_1}^{r_2} \\
&= GMm\left[\frac{1}{r_2} - \frac{1}{r_1}\right] = GMm\left[\frac{1}{r(t)} - \frac{1}{r(0)}\right]
\end{aligned}$$

Applying work-energy theorem:

$$\begin{aligned}
T(t) - T(0) = W(\boldsymbol{r}_0 \to \boldsymbol{r}) &\implies \frac{m}{2}v^2(t) - \frac{m}{2}v^2(0) = GMm\left[\frac{1}{r(t)} - \frac{1}{r(0)}\right] \\
&\implies v^2(0) = v^2(t) - 2GM\left[\frac{1}{r(t)} - \frac{1}{r(0)}\right]
\end{aligned}$$

For escape condition:

$$\begin{cases} r(t) \to \infty \\ v(t) = 0 \end{cases} \implies v^2(0) = \frac{2GM}{R_e}$$

Since $GM = gR_e^2$ at Earth's surface:

$$v^2(0) = \frac{2gR_e^2}{R_e} = 2gR_e$$

$$\boxed{v_{\text{escape}} = \sqrt{2gR_e}}$$

Problem 5.3

A point mass of mass m is attached at the end of the massless string of length L. It is released from $\theta = \theta_0$ with $\dot{\theta} = 0$ at $t = 0$.

Obtain the first-order differential equation for $\theta(t)$ governing the dynamics of the point mass. Assuming small angles, $\theta_0 \ll 1$, solve for $\theta(t)$.

Solution

Position and velocity vectors:

$$\boldsymbol{r}(t) = L\sin\theta(t)\hat{e}_y + L\cos\theta(t)\hat{e}_z, \qquad \boldsymbol{W}(t) = mg\hat{e}_z$$

$$\dot{\boldsymbol{r}}(t) = L\dot{\theta}(t)\cos\theta(t)\hat{e}_y - L\dot{\theta}(t)\sin\theta(t)\hat{e}_z$$

Kinetic energy:

$$\Longrightarrow T(t) = \frac{m}{2}\dot{\boldsymbol{r}}(t)\cdot\dot{\boldsymbol{r}}(t) = \frac{1}{2}mL^2\dot{\theta}^2(t)$$

Forces acting on the mass:

$$\boldsymbol{F}(t) = \boldsymbol{W}(t) + \boldsymbol{N}(t) = mg\hat{e}_z + N(t)\left[-\sin\theta(t)\hat{e}_y - \cos\theta(t)\hat{e}_z\right]$$

Work done by all forces:

$$W(\boldsymbol{r}(0)\to\boldsymbol{r}(t)) = \int_{\boldsymbol{r}(0)}^{\boldsymbol{r}(t)} \boldsymbol{F}(t)\cdot d\boldsymbol{r} = -\int_{\theta(0)}^{\theta(t)} mgL\sin\theta(t)d\theta$$

Work-energy theorem:

$$\begin{aligned}
&T(t) - T(0) = W(\boldsymbol{r}(0)\to\boldsymbol{r}(t))\\
&\Longrightarrow \frac{1}{2}mL^2\dot{\theta}^2(t) - \frac{1}{2}mL^2\dot{\theta}^2(0) = mgL[\cos\theta(t) - \cos\theta(0)]\\
&\Longrightarrow \frac{1}{2}L\dot{\theta}^2(t) = g\cos\theta(t) - g\cos\theta_0\\
&\Longrightarrow \frac{\mathrm{d}\theta}{\mathrm{d}t} = -\sqrt{2\frac{g}{L}[\cos\theta(t) - \cos\theta_0]}\\
&\Longrightarrow \sqrt{2\frac{g}{L}}\int_0^t dt = -\int_{\theta_0}^{\theta(t)} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}}\\
&\Longrightarrow \sqrt{2\frac{g}{L}}\int_0^t dt = -\sqrt{2}\int_{\theta_0}^{\theta(t)} \frac{d\theta}{\sqrt{\theta_0^2 - \theta^2}} \qquad \left[\because \cos\theta \approx 1 - \frac{\theta^2}{2}\right]\\
&\Longrightarrow \sqrt{\frac{g}{L}}\int_0^t dt = -\int_{\theta_0}^{\theta(t)} \left(\sqrt{1 - \frac{\theta^2}{\theta_0^2}}\right)^{-1} d\left(\frac{\theta}{\theta_0}\right)\\
&\Longrightarrow \sqrt{\frac{g}{L}}t = -\sin^{-1}\left(\frac{\theta(t)}{\theta_0}\right) + \frac{\pi}{2}
\end{aligned}$$

$$\boxed{\theta(t) = \theta_0\cos\left(\sqrt{\frac{g}{L}}t\right)}$$