Assignment 6

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PC3261
Classical Mechanics II

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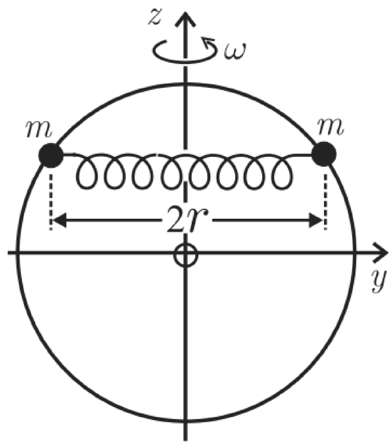
Problem 1

[40 pts] Beads on rotating hoop

Consider a system consisting of two beads, a massless spring and a circular light hoop. The beads are connected by the spring and they slide without friction on the hoop. The hoop lies in the yz -plane with its center at the origin of the coordinate system. The yz-plane is horizontal and the spring is parallel to the y-axis. Each bead has mass m, the force constant of the spring is k and the radius of the hoop is R. The equilibrium length $2r_0$ of the spring is less that the diameter of the hoop, i.e. $r_0 < R$. Suppose the hoop rotates about the z-axis with a constant angular speed ω.



(a) Express the Lagrangian in terms of cylindrical coordinates and show that it can be written in the one-dimensional form:

$$L = \frac{1}{2}\mu\dot{z}^2 - V_{\text{eff}(z)}$$

, where μ is the position-dependent effective mass and V_{eff} is the one-dimensional effective potential. Determine the expressions for μ and $V_{\text{eff}(z)}$.

(b) Determine the equilibrium points z_ω of the beads. And, determine the stability of these equilibrium points. Show that there exists a critical angular speed ω_{crit} at which the stability of these equilibrium points should depend on whether $\omega > \omega_{\text{crit}}$, $\omega = \omega_{\text{crit}}$ or $\omega < \omega_{\text{crit}}$ respectively.

(c) Now, suppose the axis of rotation of the hoop is turned through an angle α about the y-axis. Determine the effect of a uniform gravitational field $\vec{g} = -g\hat{e}_x$ on the above results.

(d) Determine the angular frequencies of small oscillations about the equilibrium points when $\alpha = 0$. Express the results in terms of ω, ω_{crit} and $\omega_0 \equiv \sqrt{2\frac{k}{m}}$.

Remark: This system is a mechanical equivalent of a thermodynamic system where the state is characterized by three physical variables: z_ω, ω and g. This simple system provides insights into topics like spontaneous symmetry breaking, phase transitions, order parameters and critical exponents in the thermodynamic system. The angular speed ω is analogous to the temperature in a thermodynamic system. The equilibrium positions z_ω play the role of an order parameter which 'spontaneously' acquires a non-zero value growing as $\sqrt{\omega_{\text{crit}} - \omega}$ just below ω_{crit}. The critical exponent for this order parameter is $\frac{1}{2}$ which is very familiar in the Landau theory for various systems with second-order phase transitions. The role of the gravitational force is to cause an explicit symmetry breaking illustrating the difference between explicit and spontaneous symmetry breaking.

Solution

Part (a): Lagrangian and effective potential

Setup and coordinates

Two beads on circular hoop in yz-plane, radius R. Hoop rotates about z-axis at constant ω.

Use cylindrical coordinates (ρ, φ, z) where:

- ρ = perpendicular distance from z-axis
- φ = azimuthal angle (increases with rotation)
- z = height along rotation axis

Constraint (on hoop): $\rho^2 + z^2 = R^2$, so $\rho = \sqrt{R^2 - z^2}$

Positions of beads

By symmetry of the rotating system and spring force, the two beads remain symmetrically placed in the rotating frame, both at the same height z but on opposite sides of the hoop (separated by angle pi):

Bead 1: $(\rho, \varphi_1, z_1) = \left(\sqrt{R^2 - z^2}, \omega t, z\right)$

Bead 2: $(\rho, \varphi_2, z_2) = \left(\sqrt{R^2 - z^2}, \omega t + \pi, z\right)$

Both beads are at the same z-coordinate, but on opposite sides of the hoop in the xy-plane.

Due to symmetry, we need only one generalized coordinate: z (position along z-axis, same for both beads).

Velocity of bead 1 in inertial frame

Position in Cartesian:

$$x_1 = \rho \cos(\omega t) = \sqrt{R^2 - z^2} \cos(\omega t)$$
$$y_1 = \rho \sin(\omega t) = \sqrt{R^2 - z^2} \sin(\omega t)$$
$$z_1 = z$$

Velocity:

$$\dot{x}_1 = \frac{-z\dot{z}}{\sqrt{R^2 - z^2}} \cos(\omega t) - \omega\sqrt{R^2 - z^2} \sin(\omega t)$$
$$\dot{y}_1 = \frac{-z\dot{z}}{\sqrt{R^2 - z^2}} \sin(\omega t) + \omega\sqrt{R^2 - z^2} \cos(\omega t)$$
$$\dot{z}_1 = \dot{z}$$

Speed squared:

$$v_1^2 = \dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2$$

$$= \frac{z^2\dot{z}^2}{R^2-z^2}(\cos^2(\omega t)+\sin^2(\omega t)) + \omega^2(R^2-z^2)(\sin^2(\omega t)+\cos^2(\omega t))$$

$$+\frac{z\dot{z}\omega\sqrt{R^2-z^2}}{\sqrt{R^2-z^2}}(-\cos(\omega t)\sin(\omega t)+\sin(\omega t)\cos(\omega t)) + \dot{z}^2$$

$$= \frac{z^2\dot{z}^2}{R^2-z^2} + \omega^2(R^2-z^2) + \dot{z}^2$$

$$= \dot{z}^2\left(\frac{z^2}{R^2-z^2}+1\right) + \omega^2(R^2-z^2)$$

$$= \dot{z}^2\frac{R^2}{R^2-z^2} + \omega^2(R^2-z^2)$$

Kinetic energy

By symmetry, bead 2 has the same kinetic energy:

$$T = 2\cdot\frac{1}{2}mv_1^2 = m\left[\dot{z}^2\frac{R^2}{R^2-z^2}+\omega^2(R^2-z^2)\right]$$

$$= \frac{mR^2\dot{z}^2}{R^2-z^2} + m\omega^2(R^2-z^2)$$

Potential energy

Distance between beads:

- Bead 1 at $(\rho\cos 0, \rho\sin 0, z) = (\rho, 0, z)$ in rotating frame
- Bead 2 at $(\rho\cos\pi, \rho\sin\pi, z) = (-\rho, 0, z)$ in rotating frame

Distance in 3D:

$$d^2 = (\rho-(-\rho))^2 + (0-0)^2 + (z-z)^2 = 4\rho^2 = 4(R^2-z^2)$$

$$d = 2\rho = 2\sqrt{R^2-z^2}$$

Spring potential energy:

$$V_{\text{spring}} = \frac{1}{2}k(d-2r_0)^2 = \frac{1}{2}k\left(2\sqrt{R^2-z^2}-2r_0\right)^2 = 2k\left(\sqrt{R^2-z^2}-r_0\right)^2$$

Lagrangian

$$L = T - V = \frac{mR^2\dot{z}^2}{R^2-z^2} + m\omega^2(R^2-z^2) - 2k\left(\sqrt{R^2-z^2}-r_0\right)^2$$

Rearranging:

$$L = \frac{mR^2}{R^2-z^2}\dot{z}^2 - \left[2k\left(\sqrt{R^2-z^2}-r_0\right)^2 - m\omega^2(R^2-z^2)\right]$$

This is of the form $L = \frac{1}{2}\mu(z)\dot{z}^2 - V_{\text{eff}(z)}$ where:

$$\boxed{\mu(z) = \frac{2mR^2}{R^2-z^2}}$$

$$\boxed{V_{\text{eff}(z)} = 2k\left(\sqrt{R^2-z^2}-r_0\right)^2 - m\omega^2(R^2-z^2)}$$

Simplified form

Let $\rho = \sqrt{R^2 - z^2}$. Then:

$$\boxed{\begin{gathered}\mu = \tfrac{2mR^2}{\rho^2} \\ V_{\text{eff}} = 2k(\rho - r_0)^2 - m\omega^2\rho^2\end{gathered}}$$

Part (b): Equilibrium points and stability

Equation of motion

The Euler-Lagrange equation with z-dependent mass $\mu(z)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mu(z)\dot{z}) = \frac{\mathrm{d}L}{\mathrm{d}z}$$

At equilibrium: $\dot{z} = 0$ and $\frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}z} = 0$

Using $V_{\text{eff}} = 2k\left(\sqrt{R^2 - z^2} - r_0\right)^2 - m\omega^2(R^2 - z^2)$:

$$\begin{aligned}\frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}z} &= 2k \cdot 2\left(\sqrt{R^2 - z^2} - r_0\right) \cdot \frac{-z}{\sqrt{R^2 - z^2}} - m\omega^2 \cdot (-2z) \\ &= -\frac{4kz}{\sqrt{R^2 - z^2}}\left(\sqrt{R^2 - z^2} - r_0\right) + 2m\omega^2 z \\ &= z\left[-\frac{4k(R^2 - z^2) - 4kr_0\sqrt{R^2 - z^2}}{\sqrt{R^2 - z^2}} + 2m\omega^2\right] \\ &= z\left[-\frac{4k\sqrt{R^2 - z^2} - (4kr_0)}{\sqrt{R^2 - z^2}} + 2m\omega^2\right]\end{aligned}$$

For equilibrium, either $z = 0$ or the bracket vanishes.

Equilibrium 1: $z = 0$ (equatorial plane)

At $z = 0$: $\rho = R$, distance $d = 2R$

Using $\rho = \sqrt{R^2 - z^2}$:

$$\frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}z} = \frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}\rho} \cdot \frac{\mathrm{d}\rho}{\mathrm{d}z} = \frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}\rho} \cdot \frac{-z}{\sqrt{R^2 - z^2}} = -\frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}\rho} \cdot \frac{z}{\rho}$$

So equilibrium when $z = 0$ or $\frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}\rho} = 0$.

$$\begin{aligned}\frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}\rho} &= 2k \cdot 2(\rho - r_0) - 2m\omega^2\rho \\ &= 4k(\rho - r_0) - 2m\omega^2\rho \\ &= 4k\rho - 4kr_0 - 2m\omega^2\rho \\ &= 2\rho(2k - m\omega^2) - 4kr_0\end{aligned}$$

Setting to zero:

$$2\rho(2k - m\omega^2) = 4kr_0$$

$$\boxed{\rho_\omega = \frac{2kr_0}{2k - m\omega^2}}$$

This is physical (positive) only if $2k - m\omega^2 > 0$, i.e., $\omega < \sqrt{2\frac{k}{m}} = \omega_0$.

Also need $\rho_\omega \leq R$:

$$\frac{2kr_0}{2k - m\omega^2} \leq R$$

$$2kr_0 \leq R(2k - m\omega^2)$$

$$m\omega^2 R \leq 2Rk - 2kr_0$$

$$\omega^2 \leq \frac{2k}{m}\left(1 - \frac{r_0}{R}\right)$$

Define critical frequency:

$$\boxed{\omega_{\text{crit}} = \sqrt{\frac{2k}{m}\left(1 - \frac{r_0}{R}\right)} = \omega_0\sqrt{1 - \frac{r_0}{R}}}$$

where $\omega_0 = \sqrt{2\frac{k}{m}}$ as given in part (d).

Summary of equilibria:

1. $\boldsymbol{z = 0}$ (beads at equator): Always an equilibrium
2. $\boldsymbol{\rho = \rho_\omega}$ (equivalently, $z = \pm\sqrt{R^2 - \rho_\omega^2}$): Exists only for $\omega < \omega_{\text{crit}}$

Stability analysis

Use second derivative test. At $z = 0$ (equivalently $\rho = R$):

$$\frac{\mathrm{d}^2 V_{\text{eff}}}{\mathrm{d}\rho^2}|_{\rho=R} = 4k - 2m\omega^2 = 2m\left(2\frac{k}{m} - \omega^2\right) = 2m(\omega_0^2 - \omega^2)$$

Since $\frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}z} = -\frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}\rho} \cdot \frac{z}{\rho}$, stability in z requires $\frac{\mathrm{d}^2 V_{\text{eff}}}{\mathrm{d}\rho^2} > 0$ at $\rho = R$.

- If $\omega < \omega_0 = \sqrt{2\frac{k}{m}}$: $\frac{\mathrm{d}^2 V_{\text{eff}}}{\mathrm{d}\rho^2} > 0 \to$ **stable**
- If $\omega > \omega_0 = \sqrt{2\frac{k}{m}}$: $\frac{\mathrm{d}^2 V_{\text{eff}}}{\mathrm{d}\rho^2} < 0 \to$ **unstable**

Note that $\omega_{\text{crit}} = \omega_0\sqrt{1 - \frac{r_0}{R}} < \omega_0$.

At $\rho = \rho_\omega$ (when it exists for $\omega < \omega_{\text{crit}}$):

$$\frac{\mathrm{d}^2 V_{\text{eff}}}{\mathrm{d}\rho^2} = 4k - 2m\omega^2 = 2m(\omega_0^2 - \omega^2)$$

This is the same expression, which is always positive for $\omega < \omega_0$.

The effective potential in ρ is:

$$V_{\text{eff}(\rho)} = 2k(\rho - r_0)^2 - m\omega^2\rho^2$$

This is valid only for $0 \leq \rho \leq R$. The equilibrium at $z = 0$ corresponds to $\rho = R$, which is a **boundary** of the allowed region.

For $\omega < \omega_{\text{crit}}$, there exists an interior equilibrium at $\rho = \rho_\omega < R$. At this point $\frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}\rho} = 0$ and $\frac{\mathrm{d}^2 V_{\text{eff}}}{\mathrm{d}\rho^2} > 0$, so it's a stable minimum. The boundary point $\rho = R$ then becomes unstable.

For $\omega > \omega_{\text{crit}}$, there's no interior equilibrium ($\rho_\omega > R$ would be unphysical), so $\rho = R$ is the only equilibrium and it's stable.

Equilibrium positions:

For all ω: $z = 0$ (equator, $\rho = R$)

For $\omega < \omega_{\text{crit}}$: $z = \pm z_\omega$ where $z_\omega = \sqrt{R^2 - \rho_\omega^2}$ and $\rho_\omega = \frac{2kr_0}{2k - m\omega^2}$

Stability:

For $\omega < \omega_{\text{crit}}$:

- $z = 0$: unstable (interior minimum exists)
- $z = \pm z_\omega$: stable

For $\omega > \omega_{\text{crit}}$:

- $z = 0$: stable (only equilibrium)

Critical frequency: $\omega_{\text{crit}} = \omega_0 \sqrt{1 - \frac{r_0}{R}}$ where $\omega_0 = \sqrt{2\frac{k}{m}}$

This exhibits **spontaneous symmetry breaking**: below ω_{crit}, the symmetric configuration ($z = 0$) becomes unstable, and beads move to asymmetric positions $\pm z_\omega$.

Part (c): Effect of gravity with tilted axis

When rotation axis is tilted by angle α about y-axis, with gravity $\vec{g} = -g\hat{x}$:

Original setup: z-axis is rotation axis After tilt: rotation axis makes angle α with vertical z-axis

In the tilted frame, gravity has components both parallel and perpendicular to the rotation axis.

Gravitational potential energy for the two beads:

- Bead 1: height $h_1 = z\cos\alpha - \rho\cos(\omega t)\sin\alpha$
- Bead 2: height $h_2 = -z\cos\alpha - \rho\cos(\omega t + \pi)\sin\alpha = -z\cos\alpha + \rho\cos(\omega t)\sin\alpha$

Total gravitational PE:

$$V_{\text{grav}} = mg(h_1 + h_2) = mg \cdot 0 = 0$$

With $\vec{g} = -g\hat{x}$, the gravitational potential is $V = mgx$ for each bead.

For bead 1: $x_1 = \rho\cos(\omega t)$ For bead 2: $x_2 = \rho\cos(\omega t + \pi) = -\rho\cos(\omega t)$

So: $V_{\text{grav}} = mg(x_1 + x_2) = mg \cdot 0 = 0$

The gravitational contributions cancel due to symmetry!

However, if the axis is tilted through angle α about y-axis:

- New z'-axis (rotation axis) is at angle α from z-axis
- Gravity is in $-x$ direction

The component of gravity along the rotation axis creates an effective force that breaks the symmetry.

After tilt, in the rotating frame aligned with the new axis:

$$V_{\text{grav}} = mg(x_1 + x_2)$$

where positions depend on α.

Effect of tilted axis with gravity:

When $\alpha \neq 0$, gravity creates time-dependent perturbations in the rotating frame. The effective potential gains a term proportional to $2mg\rho\sin\alpha\cos(\omega t)$, breaking rotational symmetry and causing forced oscillations at frequency ω and parametric resonance effects.

For $\alpha = 0$ (original problem), gravitational effects cancel by symmetry.

Part (d): Frequencies of small oscillations ($\alpha = 0$)

Case 1: $\omega > \omega_{\text{crit}}$ (equilibrium at $z = 0$)

Near $z = 0$, expand $V_{\text{eff}(z)}$ to second order:

$$V_{\text{eff}(z)} \approx V_{\text{eff}(0)} + \frac{1}{2}V_{\text{eff}''}(0)z^2$$

From $V_{\text{eff}(\rho)} = 2k(\rho - r_0)^2 - m\omega^2\rho^2$ with $\rho = \sqrt{R^2 - z^2}$:

At $z = 0$: $\rho = R$

$$\frac{\mathrm{d}\rho}{\mathrm{d}z}|_{z=0} = -\frac{z}{\sqrt{R^2 - z^2}}|_{z=0} = 0$$

$$\frac{\mathrm{d}^2\rho}{\mathrm{d}z^2}|_{z=0} = \frac{\mathrm{d}}{\mathrm{d}z}\left(-\frac{z}{\sqrt{R^2 - z^2}}\right) = -\left(\frac{1}{\sqrt{R^2 - z^2}} + \frac{z^2}{(R^2 - z^2)^{\frac{3}{2}}}\right)|_{z=0} = -\frac{1}{R}$$

So:

$$\frac{\mathrm{d}^2 V_{\text{eff}}}{\mathrm{d}z^2}|_{z=0} = \frac{\mathrm{d}^2 V_{\text{eff}}}{\mathrm{d}\rho^2}\left(\frac{\mathrm{d}\rho}{\mathrm{d}z}\right)^2 + \frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}\rho}\frac{\mathrm{d}^2\rho}{\mathrm{d}z^2}$$

$$= \frac{\mathrm{d}^2 V_{\text{eff}}}{\mathrm{d}\rho^2} \cdot 0 + \frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}\rho}|_{\rho=R} \cdot \left(-\frac{1}{R}\right)$$

At equilibrium $z = 0$:

$$\frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}z}|_{z=0} = -\frac{z}{\rho} \cdot \frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}\rho}|_{\rho=R} = 0$$

This is automatically zero regardless of $\frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}\rho}|_{\rho=R}$.

For the second derivative, use:

$$\frac{\mathrm{d}^2 V_{\text{eff}}}{\mathrm{d}z^2} = \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}\rho}\frac{\mathrm{d}\rho}{\mathrm{d}z}\right)$$

At $z = 0$:

$$\frac{\mathrm{d}^2 V_{\text{eff}}}{\mathrm{d}z^2}|_{z=0} = \left[\frac{\mathrm{d}^2 V_{\text{eff}}}{\mathrm{d}\rho^2}\left(\frac{\mathrm{d}\rho}{\mathrm{d}z}\right)^2 + \frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}\rho}\frac{\mathrm{d}^2\rho}{\mathrm{d}z^2}\right]|_{z=0}$$

$$= 0 + \frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}\rho}|_{\rho=R} \cdot \left(-\frac{1}{R}\right)$$

$$= -\frac{4kR - 4kr_0 - 2m\omega^2 R}{R}$$

$$= -4k + \frac{4kr_0}{R} + 2m\omega^2$$

$$= 2m\omega^2 + \frac{4kr_0}{R} - 4k$$

$$= 2m\left[\omega^2 - \frac{2k}{m(1-\frac{r_0}{R})}\right]$$

$$= 2m(\omega^2 - \omega_{\text{crit}}^2)$$

The equation of motion near $z = 0$ with $\mu(0) = 2m\frac{R^2}{R^2} = 2m$:

$$2m\ddot{z} \approx -2m(\omega^2 - \omega_{\text{crit}}^2)z$$

$$\ddot{z} + (\omega^2 - \omega_{\text{crit}}^2)z = 0$$

For $\omega > \omega_{\text{crit}}$:

$$\boxed{\Omega_1 = \sqrt{\omega^2 - \omega_{\text{crit}}^2}}$$

Case 2: $\omega < \omega_{\text{crit}}$ (equilibrium at $z = z_\omega$)

At the off-equator equilibrium $\rho = \rho_\omega$, where $\frac{dV_{\text{eff}}}{d\rho} = 0$:

The curvature is:

$$\frac{d^2V_{\text{eff}}}{d\rho^2}|_{\rho_\omega} = 4k - 2m\omega^2 = 2m\left(2\frac{k}{m} - \omega^2\right) = 2m(\omega_0^2 - \omega^2)$$

Converting to z-coordinate with $z_\omega = \sqrt{R^2 - \rho_\omega^2}$:

$$\mu(z_\omega) = \frac{2mR^2}{\rho_\omega^2}$$

The effective restoring force:

$$\frac{d^2V_{\text{eff}}}{dz^2}|_{z_\omega} = \left(\frac{z_\omega}{\rho_\omega}\right)^2 \frac{d^2V_{\text{eff}}}{d\rho^2}|_{\rho_\omega} = \left(\frac{z_\omega}{\rho_\omega}\right)^2 \cdot 2m(\omega_0^2 - \omega^2)$$

Equation of motion:

$$\frac{2mR^2}{\rho_\omega^2}\ddot{z} = -\left(\frac{z_\omega}{\rho_\omega}\right)^2 \cdot 2m(\omega_0^2 - \omega^2)z$$

$$\ddot{z} = -\frac{z_\omega^2}{R^2}(\omega_0^2 - \omega^2)z$$

$$\ddot{z} + \frac{R^2 - \rho_\omega^2}{R^2}(\omega_0^2 - \omega^2)z = 0$$

Since $z_\omega^2 = R^2 - \rho_\omega^2$:

$$\ddot{z} + \frac{z_\omega^2}{R^2}(\omega_0^2 - \omega^2)z = 0$$

For small oscillations about $z = z_\omega$, the frequency is:

$$\Omega^2 = \frac{z_\omega^2}{R^2}(\omega_0^2 - \omega^2)$$

At $\rho = \rho_\omega$, the second derivative in ρ coordinates gives the oscillation frequency directly.

Oscillation frequencies ($\alpha = 0$):

For $\omega > \omega_{\text{crit}}$ (equilibrium at $z = 0$):

$$\Omega = \sqrt{\omega^2 - \omega_{\text{crit}}^2}$$

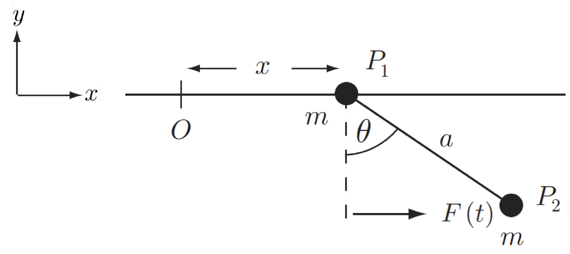
For $\omega < \omega_{\text{crit}}$ (equilibrium at $z = \pm z_\omega$):

$$\Omega = \sqrt{\omega_0^2 - \omega^2}$$

where $\omega_0 = \sqrt{2\frac{k}{m}}$ and $\omega_{\text{crit}} = \omega_0\sqrt{1 - \frac{r_0}{R}}$

Problem 2

[30 pts] Particles on rail



A system consists of two identical particles P_1 and P_2 of mass m connected by a light inextensible string of length a. The particle P_1 is constrained to move along a fixed smooth horizontal rail and the whole system moves under uniform gravity in the vertical plane through the rail.

(a) Using x and θ as generalized coordinates, find the Hamiltonian of the system.

(b) Hence, obtain differential equations for x and θ governing the dynamics of the system.

Solution

Part (a): Hamiltonian

Setup and coordinates

- Particle P_1: mass m, on horizontal rail at position x
- Particle P_2: mass m, hangs from P_1 by string length a
- String makes angle θ from vertical

Choose origin on the rail at initial position of P_1, y-axis pointing downward.

Positions

P_1: $(x, 0)$

P_2: $(x + a\sin\theta, a\cos\theta)$

Velocities

P_1: $(\dot{x}, 0)$

P_2: $\left(\dot{x} + a\dot{\theta}\cos\theta, -a\dot{\theta}\sin\theta\right)$

Kinetic energy

$$\begin{aligned}
T &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\left[\left(\dot{x} + a\dot{\theta}\cos\theta\right)^2 + a^2\dot{\theta}^2\sin^2\theta\right] \\
&= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\left[\dot{x}^2 + 2\dot{x}a\dot{\theta}\cos\theta + a^2\dot{\theta}^2\cos^2\theta + a^2\dot{\theta}^2\sin^2\theta\right] \\
&= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\left[\dot{x}^2 + 2\dot{x}a\dot{\theta}\cos\theta + a^2\dot{\theta}^2\right] \\
&= m\dot{x}^2 + m\dot{x}a\dot{\theta}\cos\theta + \frac{1}{2}ma^2\dot{\theta}^2
\end{aligned}$$

Potential energy

Taking the rail as reference ($y = 0$):

$$V = 0 + mga\cos\theta = mga\cos\theta$$

Lagrangian

$$L = T - V = m\dot{x}^2 + m\dot{x}a\dot{\theta}\cos\theta + \frac{1}{2}ma^2\dot{\theta}^2 - mga\cos\theta$$

Canonical momenta

$$p_x = \frac{\mathrm{d}L}{\mathrm{d}\dot{x}} = 2m\dot{x} + ma\dot{\theta}\cos\theta$$

$$p_\theta = \frac{\mathrm{d}L}{\mathrm{d}\dot{\theta}} = ma\dot{x}\cos\theta + ma^2\dot{\theta}$$

Invert to find velocities

From the first equation:

$$\dot{x} = \frac{p_x - ma\dot{\theta}\cos\theta}{2m}$$

Substitute into the second:

$$p_\theta = ma\cos\theta \cdot \frac{p_x - ma\dot{\theta}\cos\theta}{2m} + ma^2\dot{\theta}$$

$$p_\theta = \frac{a\cos\theta}{2}p_x - \frac{ma^2\cos^2\theta}{2}\dot{\theta} + ma^2\dot{\theta}$$

$$p_\theta = \frac{a\cos\theta}{2}p_x + ma^2\dot{\theta}\left(1 - \frac{\cos^2\theta}{2}\right)$$

$$p_\theta = \frac{a\cos\theta}{2}p_x + ma^2\dot{\theta}\left(\frac{2 - \cos^2\theta}{2}\right)$$

$$p_\theta - \frac{a\cos\theta}{2}p_x = \frac{ma^2}{2}(2 - \cos^2\theta)\dot{\theta}$$

$$\boxed{\dot{\theta} = \frac{2\left(p_\theta - \frac{a\cos\theta}{2}p_x\right)}{ma^2(2 - \cos^2\theta)} = \frac{2p_\theta - ap_x\cos\theta}{ma^2(2 - \cos^2\theta)}}$$

And:

$$\dot{x} = \frac{p_x}{2m} - \frac{a\cos\theta}{2m}\dot{\theta}$$

$$= \frac{p_x}{2m} - \frac{a\cos\theta}{2m} \cdot \frac{2p_\theta - ap_x\cos\theta}{ma^2(2 - \cos^2\theta)}$$

$$= \frac{p_x}{2m} - \frac{\cos\theta(2p_\theta - ap_x\cos\theta)}{2ma(2 - \cos^2\theta)}$$

Defining $\Delta = 2 - \cos^2\theta$:

$$\boxed{\dot{x} = \frac{p_x\Delta - ap_\theta\cos\theta}{2m\Delta}}$$

Hamiltonian

$$H = p_x\dot{x} + p_\theta\dot{\theta} - L$$

Using matrix inversion for the coupled momenta:

The kinetic energy in matrix form:

$$T = \frac{1}{2}\left[\dot{x}, \dot{\theta}\right]\begin{pmatrix} 2m & ma\cos\theta \\ ma\cos\theta & ma^2 \end{pmatrix}\begin{pmatrix} \dot{x} \\ \dot{\theta} \end{pmatrix}$$

The mass matrix:

$$M = \begin{pmatrix} 2m & ma\cos\theta \\ ma\cos\theta & ma^2 \end{pmatrix}$$

Its inverse:

$$\det(M) = 2m \cdot ma^2 - (ma\cos\theta)^2 = m^2a^2(2-\cos^2\theta) = m^2a^2\Delta$$

$$M^{-1} = \frac{1}{m^2a^2\Delta}\begin{pmatrix} ma^2 & -ma\cos\theta \\ -ma\cos\theta & 2m \end{pmatrix}$$

$$= \frac{1}{m\Delta}\begin{pmatrix} 1 & -\frac{\cos\theta}{a} \\ -\frac{\cos\theta}{a} & \frac{2}{a^2} \end{pmatrix}$$

The Hamiltonian:

$$H = \frac{1}{2}[p_x, p_\theta]M^{-1}\begin{pmatrix} p_x \\ p_\theta \end{pmatrix} + V$$

$$= \frac{1}{2m\Delta}[p_x, p_\theta]\begin{pmatrix} 1 & -\frac{\cos\theta}{a} \\ -\frac{\cos\theta}{a} & \frac{2}{a^2} \end{pmatrix}\begin{pmatrix} p_x \\ p_\theta \end{pmatrix} + mga\cos\theta$$

$$= \frac{1}{2m\Delta}\left[p_x - \frac{p_\theta\cos\theta}{a}, -\frac{p_x\cos\theta}{a} + \frac{2p_\theta}{a^2}\right]\begin{pmatrix} p_x \\ p_\theta \end{pmatrix} + mga\cos\theta$$

$$= \frac{1}{2m\Delta}\left[\left(p_x - \frac{p_\theta\cos\theta}{a}\right)p_x + \left(-\frac{p_x\cos\theta}{a} + \frac{2p_\theta}{a^2}\right)p_\theta\right] + mga\cos\theta$$

$$= \frac{1}{2m\Delta}\left[p_x^2 - \frac{p_xp_\theta\cos\theta}{a} - \frac{p_xp_\theta\cos\theta}{a} + \frac{2p_\theta^2}{a^2}\right] + mga\cos\theta$$

$$= \frac{1}{2m\Delta}\left[p_x^2 - \frac{2p_xp_\theta\cos\theta}{a} + \frac{2p_\theta^2}{a^2}\right] + mga\cos\theta$$

$$\boxed{H = \frac{p_x^2 - \frac{2p_xp_\theta\cos\theta}{a} + \frac{2p_\theta^2}{a^2}}{2m(2-\cos^2\theta)} + mga\cos\theta}$$

Or equivalently:

$$\boxed{H = \frac{a^2p_x^2 - 2ap_xp_\theta\cos\theta + 2p_\theta^2}{2ma^2(2-\cos^2\theta)} + mga\cos\theta}$$

Part (b): Equations of motion

Using Hamilton's equations:

$$\dot{x} = \frac{\mathrm{d}H}{\mathrm{d}p_x}, \quad \dot{\theta} = \frac{\mathrm{d}H}{\mathrm{d}p_\theta}$$

$$\dot{p}_x = -\frac{\mathrm{d}H}{\mathrm{d}x}, \quad \dot{p}_\theta = -\frac{\mathrm{d}H}{\mathrm{d}\theta}$$

Calculate $\dot{x}$:

$$\dot{x} = \frac{\mathrm{d}}{\mathrm{d}p_x}\left[\frac{a^2p_x^2 - 2ap_xp_\theta\cos\theta + 2p_\theta^2}{2ma^2\Delta}\right]$$

$$= \frac{2a^2p_x - 2ap_\theta\cos\theta}{2ma^2\Delta} = \frac{ap_x - p_\theta\cos\theta}{ma\Delta}$$

$$\boxed{\dot{x} = \frac{ap_x - p_\theta\cos\theta}{ma(2-\cos^2\theta)}}$$

Calculate $\dot{\theta}$:

$$\dot{\theta} = \frac{\mathrm{d}}{\mathrm{d}p_\theta}\left[\frac{a^2p_x^2 - 2ap_xp_\theta\cos\theta + 2p_\theta^2}{2ma^2\Delta}\right]$$

$$= \frac{-2ap_x\cos\theta + 4p_\theta}{2ma^2\Delta} = \frac{2p_\theta - ap_x\cos\theta}{ma^2\Delta}$$

$$\boxed{\dot{\theta} = \frac{2p_\theta - ap_x\cos\theta}{ma^2(2-\cos^2\theta)}}$$

Calculate $\dot{p}_x$:

Since H has no explicit x dependence:

$$\boxed{\dot{p}_x = -\frac{\mathrm{d}H}{\mathrm{d}x} = 0}$$

This means p_x is conserved (horizontal momentum conservation).

Calculate $\dot{p}_\theta$:

$$\dot{p}_\theta = -\frac{\mathrm{d}H}{\mathrm{d}\theta}$$

From the Lagrangian: $\frac{\mathrm{d}L}{\mathrm{d}x} = 0$ (no explicit x dependence), so:

$$\boxed{\dot{p}_x = 0 \implies p_x = \text{const}}$$

And:

$$\frac{\mathrm{d}L}{\mathrm{d}\theta} = -m\dot{x}a\dot{\theta}\sin\theta + mga\sin\theta = ma\sin\theta\left(g - \dot{x}\dot{\theta}\right)$$

$$\boxed{\dot{p}_\theta = ma\sin\theta\left(g - \dot{x}\dot{\theta}\right)}$$

Summary of equations:

From Hamilton's equations or equivalently from Lagrangian formulation:

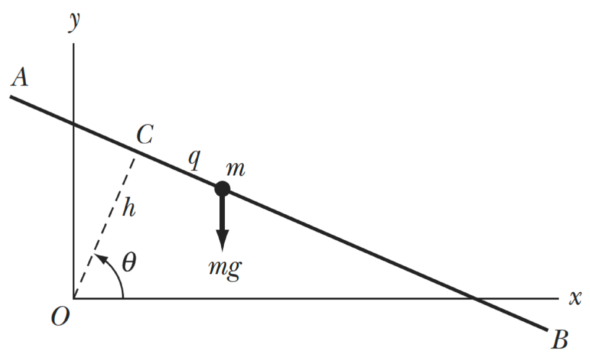
$$\boxed{\begin{aligned} \dot{x} &= \frac{ap_x - p_\theta \cos\theta}{ma(2-\cos^2\theta)} \\ \dot{\theta} &= \frac{2p_\theta - ap_x\cos\theta}{ma^2(2-\cos^2\theta)} \\ \dot{p}_x &= 0 \\ \dot{p}_\theta &= ma\sin\theta\Big(g - \dot{x}\dot{\theta}\Big) \end{aligned}}$$

Or in terms of x, θ and their derivatives:

$$\boxed{\begin{gathered} 2\ddot{x} + a\ddot{\theta}\cos\theta - a\dot{\theta}^2\sin\theta = 0 \\ \ddot{x}\cos\theta + a\ddot{\theta} = g\sin\theta \end{gathered}}$$

Problem 3

[30 pts] Particle on rotating wire



A particle of mass m can slide freely along a light wire AB whose perpendicular distance to the origin O is h. The line OC rotates about the origin at a constant angular speed Ω. The position of the particle can be described in terms of the angle θ and the distance q to the point C. The initial conditions are $\theta(0) = 0$, $q(0) = 0$ and $\dot{q}(0) = 0$.

(a) Using q as the generalized coordinate, find the Hamiltonian of the system.

(b) Is the total mechanical energy given by the Hamiltonian? Is the total mechanical energy conserved? Explain.

(c) Obtain equation of motion using Hamilton equation of motion. Solve for $q(t)$.

Solution

Part (a): Hamiltonian

Setup and coordinates

- Wire AB at perpendicular distance h from origin O
- Point C is the foot of perpendicular from O to wire
- Line OC rotates at constant Ω, so $\theta(t) = \Omega t$ (with $\theta(0) = 0$)
- Particle at distance q from C along wire

Position in Cartesian coordinates

In the fixed (inertial) frame, with x-axis along initial direction of OC:

- Point C rotates: $(h\cos(\Omega t), h\sin(\Omega t))$
- Particle is at distance q from C along the wire
- Wire is perpendicular to OC, so along direction $(-\sin(\Omega t), \cos(\Omega t))$

Position vector:

$$\vec{r} = (h\cos(\Omega t) + q\sin(\Omega t))\hat{x} + (h\sin(\Omega t) - q\cos(\Omega t))\hat{y}$$

Velocity

$$\begin{aligned}\dot{\vec{r}} &= [(-h\Omega\sin(\Omega t) + \dot{q}\sin(\Omega t) + q\Omega\cos(\Omega t))]\hat{x} \\ &\quad + [(h\Omega\cos(\Omega t) - \dot{q}\cos(\Omega t) - q\Omega\sin(\Omega t))]\hat{y} \\ &= [(\dot{q} - h\Omega)\sin(\Omega t) + q\Omega\cos(\Omega t)]\hat{x} \\ &\quad + [(\dot{q} - h\Omega)\cos(\Omega t) - q\Omega\sin(\Omega t)]\hat{y}\end{aligned}$$

Kinetic energy

$$v^2 = \dot{\vec{r}} \cdot \dot{\vec{r}} = (\dot{q} - h\Omega)^2 + q^2\Omega^2$$

$$T = \frac{1}{2}m[(\dot{q} - h\Omega)^2 + q^2\Omega^2]$$

Potential energy

$$y = h\sin(\Omega t) - q\cos(\Omega t)$$

$$V = mgy = mg(h\sin(\Omega t) - q\cos(\Omega t))$$

Lagrangian

$$L = T - V = \frac{1}{2}m[(\dot{q} - h\Omega)^2 + q^2\Omega^2] - mg(h\sin(\Omega t) - q\cos(\Omega t))$$

Canonical momentum

$$p = \frac{\mathrm{d}L}{\mathrm{d}\dot{q}} = m(\dot{q} - h\Omega)$$

$$\dot{q} = \frac{p}{m} + h\Omega$$

Hamiltonian

$$\begin{aligned} H &= p\dot{q} - L \\ &= p\left(\frac{p}{m} + h\Omega\right) - \left[\frac{1}{2}m\left(\frac{p}{m}\right)^2 + \frac{1}{2}mq^2\Omega^2 - mg(h\sin(\Omega t) - q\cos(\Omega t))\right] \\ &= \frac{p^2}{m} + ph\Omega - \frac{p^2}{2m} - \frac{1}{2}mq^2\Omega^2 + mgh\sin(\Omega t) - mgq\cos(\Omega t) \\ &= \frac{p^2}{2m} + ph\Omega - \frac{1}{2}mq^2\Omega^2 + mgh\sin(\Omega t) - mgq\cos(\Omega t) \end{aligned}$$

$$\boxed{H(q,p,t) = \frac{p^2}{2m} + \Omega hp - \frac{1}{2}m\Omega^2 q^2 + mgh\sin(\Omega t) - mgq\cos(\Omega t)}$$

Part (b): Energy conservation

Total mechanical energy:

$$\begin{aligned} E &= T + V \\ &= \frac{1}{2}m[(\dot{q} - h\Omega)^2 + q^2\Omega^2] + mg(h\sin(\Omega t) - q\cos(\Omega t)) \end{aligned}$$

Using $p = m(\dot{q} - h\Omega)$:

$$\begin{aligned} E &= \frac{1}{2}m\left(\frac{p}{m}\right)^2 + \frac{1}{2}mq^2\Omega^2 + mg(h\sin(\Omega t) - q\cos(\Omega t)) \\ &= \frac{p^2}{2m} + \frac{1}{2}mq^2\Omega^2 + mgh\sin(\Omega t) - mgq\cos(\Omega t) \end{aligned}$$

Comparing with H:

$$E - H = \frac{p^2}{2m} + \frac{1}{2}mq^2\Omega^2 - \frac{p^2}{2m} - \Omega hp + \frac{1}{2}m\Omega^2 q^2 = m\Omega^2 q^2 - \Omega hp$$

$$\boxed{E - H = m\Omega^2 q^2 - \Omega hp \neq 0 \implies H \neq E}$$

The Hamiltonian does **not** equal the total mechanical energy.

Is energy conserved?

The Hamiltonian has explicit time dependence through $\sin(\Omega t)$:

$$\frac{\mathrm{d}H}{\mathrm{d}t} = mgh\Omega\cos(\Omega t) \neq 0$$

Therefore the Hamiltonian is **not** conserved.

The total mechanical energy E also has explicit time dependence, so it is **not** conserved.

Energy analysis:

- Total mechanical energy: $E = \frac{p^2}{2m} + \frac{1}{2}mq^2\Omega^2 + mgh\sin(\Omega t) - mgq\cos(\Omega t)$
- Hamiltonian: $H = \frac{p^2}{2m} + \Omega hp - \frac{1}{2}m\Omega^2 q^2 + mgh\sin(\Omega t) - mgq\cos(\Omega t) \neq E$
- $\frac{\mathrm{d}H}{\mathrm{d}t} = mgh\Omega\cos(\Omega t) \neq 0$: Hamiltonian is **not conserved**
- Total energy is **not conserved**

Reason: The rotating wire constraint is time-dependent (rheonomic), and external work is done to maintain the rotation. The gravitational potential also has explicit time dependence in the rotating frame.

Part (c): Equation of motion and solution

Hamilton's equations:

$$\dot{q} = \frac{\mathrm{d}H}{\mathrm{d}p} = \frac{p}{m} + h\Omega$$

$$\dot{p} = -\frac{\mathrm{d}H}{\mathrm{d}q} = m\Omega^2 q + mg\cos(\Omega t)$$

$$\begin{aligned}\ddot{q} &= \frac{\dot{p}}{m} \\ &= \Omega^2 q + g\cos(\Omega t)\end{aligned}$$

$$\boxed{\ddot{q} - \Omega^2 q = g\cos(\Omega t)}$$

General solution:

Homogeneous solution:

$$q_{h(t)} = C_1\cosh(\Omega t) + C_2\sinh(\Omega t)$$

Particular solution: $q_{p(t)} = A_p\cos(\Omega t)$

$$\ddot{q}_p = -A_p\Omega^2\cos(\Omega t)$$

$$-A_p\Omega^2\cos(\Omega t) - \Omega^2 A_p\cos(\Omega t) = g\cos(\Omega t)$$

$$A_p = -\frac{g}{2\Omega^2}$$

$$q(t) = C_1\cosh(\Omega t) + C_2\sinh(\Omega t) - \frac{g}{2\Omega^2}\cos(\Omega t)$$

Initial conditions:

$q(0) = 0$:

$$0 = C_1 - \frac{g}{2\Omega^2} \Longrightarrow C_1 = \frac{g}{2\Omega^2}$$

$\dot{q}(0) = 0$:

$$\dot{q}(t) = C_1\Omega\sinh(\Omega t) + C_2\Omega\cosh(\Omega t) + \frac{g}{2\Omega}\sin(\Omega t)$$

$$\dot{q}(0) = C_2\Omega = 0 \Longrightarrow C_2 = 0$$

$$\boxed{q(t) = \frac{g}{2\Omega^2}\left[\cosh(\Omega t) - \cos(\Omega t)\right]}$$