Assignment 5

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Classical Mechanics II

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Problem 1

[20 pts] Spring pendulum

A mass m hangs from a massless spring (constant k, natural length ℓ_0) attached to a fixed pivot. The system can move in a vertical plane under gravity.

Using suitable generalized coordinates, derive the equations of motion. Solve for small angular and radial displacements from equilibrium.

Solution

Generalized coordinates

Use polar coordinates (r, θ) where:

- r = length of spring (distance from pivot to mass)
- θ = angle from vertical (positive clockwise)

Position and velocity

Position in Cartesian coordinates (origin at pivot, y downward):

$$x = r \sin\theta, \quad y = r\cos\theta$$

Velocity:

$$\dot{x} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$$

$$\dot{y} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$$

Speed squared:

$$v^2 = \dot{x}^2 + \dot{y}^2 = \dot{r}^2(\sin^2\theta + \cos^2\theta) + r^2\dot{\theta}^2(\cos^2\theta + \sin^2\theta) = \dot{r}^2 + r^2\dot{\theta}^2$$

Lagrangian

Kinetic energy:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

Potential energy (taking pivot as reference, y positive downward):

$$V = -mgr\cos\theta + \frac{1}{2}k(r - \ell_0)^2$$

(gravitational PE is negative since mass hangs below pivot; spring PE with natural length ℓ_0)

Lagrangian:

$$L = T - V = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + mgr\cos\theta - \frac{1}{2}k(r - \ell_0)^2$$

Equations of motion

For r:

$$\frac{\mathrm{d}L}{\mathrm{d}r} = mr\dot{\theta}^2 + mg\cos\theta - k(r - \ell_0)$$

$$\frac{\mathrm{d}L}{\mathrm{d}\dot{r}} = m\dot{r}, \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}L}{\mathrm{d}\dot{r}}\right) = m\ddot{r}$$

Euler-Lagrange equation:

$$m\ddot{r} = mr\dot{\theta}^2 + mg\cos\theta - k(r - \ell_0)$$

$$\boxed{m\ddot{r} - mr\dot{\theta}^2 - mg\cos\theta + k(r - \ell_0) = 0}$$

For θ:

$$\frac{\mathrm{d}L}{\mathrm{d}\theta} = -mgr\sin\theta$$

$$\frac{\mathrm{d}L}{\mathrm{d}\dot{\theta}} = mr^2\dot{\theta}, \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}L}{\mathrm{d}\dot{\theta}}\right) = m\left(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}\right)$$

Euler-Lagrange equation:

$$m\left(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}\right) = -mgr\sin\theta$$

$$2r\dot{r}\dot{\theta} + r^2\ddot{\theta} + gr\sin\theta = 0$$

Dividing by r:

$$\boxed{r\ddot{\theta} + 2\dot{r}\dot{\theta} + g\sin\theta = 0}$$

Or equivalently:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot{\theta}\right) + gr\sin\theta = 0}$$

Equilibrium position

At equilibrium: $\dot{r} = \dot{\theta} = \ddot{r} = \ddot{\theta} = 0$ and $\theta = 0$ (hanging straight down).

From the r equation with $\theta = 0$, $\dot{\theta} = 0$:

$$0 = mg - k\left(r_{\text{eq}} - \ell_0\right)$$

$$\boxed{r_{\text{eq}} = \ell_0 + \frac{mg}{k}}$$

This is the equilibrium length: natural length plus extension due to weight.

Small oscillations about equilibrium

Let:

$$r(t) = r_{\text{eq}} + \rho(t), \quad \theta(t) = \varepsilon(t)$$

where $\rho, \varepsilon \ll 1$ (small displacements).

Linearized radial equation

Substitute $r = r_{\text{eq}} + \rho$, $\theta = \varepsilon$ into the r equation:

$$m\ddot{\rho} - m\left(r_{\text{eq}} + \rho\right)\dot{\varepsilon}^2 - mg\cos\varepsilon + k\rho = 0$$

The r equation is:

$$m\ddot{r} = mr\dot{\theta}^2 + mg\cos\theta - k(r - \ell_0)$$

At equilibrium ($r = r_{\text{eq}}$, $\theta = 0$, $\dot{\theta} = 0$):

$$0 = mg - k\left(r_{\text{eq}} - \ell_0\right)$$

So: $k\left(r_{\text{eq}} - \ell_0\right) = mg$

For small oscillations $r = r_{\text{eq}} + \rho$, $\theta = \varepsilon$:

$$m\ddot{\rho} = m\left(r_{\text{eq}} + \rho\right)\dot{\varepsilon}^2 + mg\cos\varepsilon - k\left(r_{\text{eq}} + \rho - \ell_0\right)$$

Using $k\left(r_{\text{eq}} - \ell_0\right) = mg$ and expanding to first order: $\cos\varepsilon \approx 1$, $\dot{\varepsilon}^2 \approx 0$:

$$m\ddot{\rho} \approx -k\rho$$

$$\boxed{\ddot{\rho} + \left(\frac{k}{m}\right)\rho = 0}$$

This gives radial oscillations with frequency:

$$\boxed{\omega_r = \sqrt{\frac{k}{m}}}$$

Linearized angular equation

From $r\ddot{\theta} + 2\dot{r}\dot{\theta} + g\sin\theta = 0$, substitute $r = r_{\text{eq}} + \rho$, $\theta = \varepsilon$:

$$\left(r_{\text{eq}} + \rho\right)\ddot{\varepsilon} + 2\dot{\rho}\dot{\varepsilon} + g\sin\varepsilon = 0$$

For small ε: $\sin\varepsilon \approx \varepsilon$, and dropping second-order term $\rho\ddot{\varepsilon}$ and product $\dot{\rho}\dot{\varepsilon}$:

$$r_{\text{eq}}\ddot{\varepsilon} + g\varepsilon = 0$$

$$\boxed{\ddot{\varepsilon} + \frac{g}{r_{\text{eq}}}\varepsilon = 0}$$

This gives angular oscillations with frequency:

$$\boxed{\omega_\theta = \sqrt{\frac{g}{r_{\text{eq}}}} = \sqrt{\frac{g}{\ell_0 + mg/k}} = \sqrt{\frac{gk}{k\ell_0 + mg}}}$$

Summary of small oscillation solution

The radial and angular motions decouple to first order:

Radial:

$$\rho(t) = A\cos(\omega_r t + \varphi_r), \quad \omega_r = \sqrt{\frac{k}{m}}$$

Angular:

$$\varepsilon(t) = B\cos(\omega_\theta t + \varphi_\theta), \quad \omega_\theta = \sqrt{\frac{g}{r_{\text{eq}}}}$$

where $A, B, \varphi_r, \varphi_\theta$ are determined by initial conditions.

Complete solution for small oscillations:

$$r(t) = r_{\text{eq}} + A\cos\left(\sqrt{\tfrac{k}{m}}t + \varphi_r\right)$$

$$\theta(t) = B\cos\left(\sqrt{\tfrac{g}{r_{\text{eq}}}}t + \varphi_\theta\right)$$

where $r_{\text{eq}} = \ell_0 + mg/k$

Problem 2

[20 pts] Time-dependent Lagrangian

Consider a system with Lagrangian:

$$L(q, \dot{q}, t) = e^{\gamma t}\left[\frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2\right]$$

where $\gamma, m, k > 0$ are constants.

(a) Derive the Euler-Lagrange equation. Are there any constants of motion? Describe the motion.

(b) Consider the point transformation $Q = e^{\gamma t/2}q$. Find the new Lagrangian $L'\left(Q, \dot{Q}, t\right)$ and its Euler-Lagrange equation. Are there any constants of motion? What is the relationship between solutions in the two formulations?

Solution

Part (a): Euler-Lagrange equation

Given: $L = e^{\gamma t}\left[\frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2\right]$

Compute partial derivatives:

$$\frac{\mathrm{d}L}{\mathrm{d}q} = e^{\gamma t} \cdot (-kq) = -kqe^{\gamma t}$$

$$\frac{\mathrm{d}L}{\mathrm{d}\dot{q}} = e^{\gamma t} \cdot m\dot{q} = m\dot{q}e^{\gamma t}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}L}{\mathrm{d}\dot{q}}\right) = m\ddot{q}e^{\gamma t} + m\dot{q} \cdot \gamma e^{\gamma t} = e^{\gamma t}(m\ddot{q} + \gamma m\dot{q})$$

Euler-Lagrange equation:

$$e^{\gamma t}(m\ddot{q} + \gamma m\dot{q}) = -kqe^{\gamma t}$$

Dividing by $e^{\gamma t}$:

$$m\ddot{q} + \gamma m\dot{q} + kq = 0$$

$$\boxed{\ddot{q} + \gamma\dot{q} + \omega_0^2 q = 0}$$

where $\omega_0^2 = \frac{k}{m}$.

This is a damped harmonic oscillator equation with damping coefficient γ.

Constants of motion

Check if energy is conserved. The Jacobi energy (generalized energy) is:

$$h = \dot{q}\frac{\mathrm{d}L}{\mathrm{d}\dot{q}} - L = \dot{q} \cdot m\dot{q}e^{\gamma t} - e^{\gamma t}\left[\frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2\right]$$

$$= e^{\gamma t}\left[m\dot{q}^2 - \frac{1}{2}m\dot{q}^2 + \frac{1}{2}kq^2\right] = e^{\gamma t}\left[\frac{1}{2}m\dot{q}^2 + \frac{1}{2}kq^2\right]$$

Taking time derivative:

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \gamma e^{\gamma t}\left[\frac{1}{2}m\dot{q}^2 + \frac{1}{2}kq^2\right] + e^{\gamma t}[m\dot{q}\ddot{q} + kq\dot{q}]$$

Using the equation of motion $m\ddot{q} = -\gamma m\dot{q} - kq$:

$$\begin{aligned}\frac{\mathrm{d}h}{\mathrm{d}t} &= \gamma e^{\gamma t}\left[\frac{1}{2}m\dot{q}^2 + \frac{1}{2}kq^2\right] + e^{\gamma t}\dot{q}[-\gamma m\dot{q} - kq + kq] \\ &= \gamma e^{\gamma t}\left[\frac{1}{2}m\dot{q}^2 + \frac{1}{2}kq^2\right] - \gamma m\dot{q}^2 e^{\gamma t} \\ &= \gamma e^{\gamma t}\left[\frac{1}{2}m\dot{q}^2 + \frac{1}{2}kq^2 - m\dot{q}^2\right] \\ &= \gamma e^{\gamma t}\left[-\frac{1}{2}m\dot{q}^2 + \frac{1}{2}kq^2\right] \neq 0\end{aligned}$$

Therefore, the Jacobi energy is NOT conserved.

Since $\frac{\mathrm{d}L}{\mathrm{d}t} \neq 0$ (explicit time dependence), energy is not conserved.

$$\boxed{\textbf{No constants of motion} \text{ (for generic initial conditions)}}$$

Description of motion

The equation $\ddot{q} + \gamma\dot{q} + \omega_0^2 q = 0$ describes a damped harmonic oscillator.

Define the discriminant: $\Delta = \gamma^2 - 4\omega_0^2$

Case 1: Underdamped ($\gamma^2 < 4\omega_0^2$)

Characteristic equation: $r^2 + \gamma r + \omega_0^2 = 0$

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}}{2} = -\frac{\gamma}{2} \pm i\omega_d$$

where $\omega_d = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$ is the damped frequency.

Solution:

$$q(t) = e^{-\gamma t/2}(A\cos(\omega_d t) + B\sin(\omega_d t))$$

$$\boxed{q(t) = e^{-\gamma t/2} C\cos(\omega_d t - \varphi)}$$

where C and φ are determined by initial conditions.

Case 2: Critically damped ($\gamma^2 = 4\omega_0^2$)

$$r = -\frac{\gamma}{2}$$

(repeated root)

Solution:

$$\boxed{q(t) = (A + Bt)e^{-\gamma t/2}}$$

Case 3: Overdamped ($\gamma^2 > 4\omega_0^2$)

$$r = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2}$$

Solution:

$$\boxed{q(t) = Ae^{r_1 t} + Be^{r_2 t}}$$

where both $r_1, r_2 < 0$ (exponential decay).

Physical interpretation: The time-dependent factor $e^{\gamma t}$ in the Lagrangian acts as a gauge transformation. The actual motion is that of a damped oscillator, with amplitude decaying as $e^{-\gamma t/2}$ (for underdamped case).

Part (b): Point transformation

Given transformation: $Q = e^{\gamma t/2} q$

Inverse: $q = e^{-\gamma t/2} Q$

Velocity transformation:

$$\begin{aligned}\dot{q} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-\gamma t/2} Q\right) = e^{-\gamma t/2}\dot{Q} + Q \cdot \left(-\frac{\gamma}{2}\right) e^{-\gamma t/2} \\ &= e^{-\gamma t/2}\left(\dot{Q} - \frac{\gamma}{2} Q\right)\end{aligned}$$

New Lagrangian

Substitute into original Lagrangian:

$$\begin{aligned}L &= e^{\gamma t}\left[\frac{1}{2} m\dot{q}^2 - \frac{1}{2} k q^2\right] \\ &= e^{\gamma t}\left[\frac{1}{2} m e^{-\gamma t}\left(\dot{Q} - \frac{\gamma}{2} Q\right)^2 - \frac{1}{2} k e^{-\gamma t} Q^2\right] \\ &= e^{\gamma t} \cdot e^{-\gamma t}\left[\frac{1}{2} m\left(\dot{Q} - \frac{\gamma}{2} Q\right)^2 - \frac{1}{2} k Q^2\right] \\ &= \frac{1}{2} m\left(\dot{Q} - \frac{\gamma}{2} Q\right)^2 - \frac{1}{2} k Q^2\end{aligned}$$

Expanding $\left(\dot{Q} - \frac{\gamma}{2} Q\right)^2 = \dot{Q}^2 - \gamma\dot{Q}Q + \frac{\gamma^2}{4} Q^2$:

$$\begin{aligned}L' &= \frac{1}{2} m\dot{Q}^2 - \frac{1}{2} m\gamma\dot{Q}Q + \frac{1}{2} m\frac{\gamma^2}{4} Q^2 - \frac{1}{2} k Q^2 \\ &= \frac{1}{2} m\dot{Q}^2 - \frac{1}{2} m\gamma\dot{Q}Q + \frac{1}{2}\left(m\frac{\gamma^2}{4} - k\right) Q^2\end{aligned}$$

$$\boxed{L'\left(Q, \dot{Q}\right) = \frac{1}{2} m\dot{Q}^2 - \frac{m\gamma}{2}\dot{Q}Q - \frac{1}{2}\left(k - \frac{m\gamma^2}{4}\right) Q^2}$$

Note: This Lagrangian does NOT have explicit time dependence!

Euler-Lagrange equation in new coordinates

$$\frac{\mathrm{d}L'}{\mathrm{d}Q} = -\frac{m\gamma}{2}\dot{Q} - \left(k - \frac{m\gamma^2}{4}\right) Q$$

$$\frac{\mathrm{d}L'}{\mathrm{d}\dot{Q}} = m\dot{Q} - \frac{m\gamma}{2} Q$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}L'}{\mathrm{d}\dot{Q}}\right) = m\ddot{Q} - \frac{m\gamma}{2}\dot{Q}$$

Euler-Lagrange equation:

$$m\ddot{Q} - \frac{m\gamma}{2}\dot{Q} = -\frac{m\gamma}{2}\dot{Q} - \left(k - \frac{m\gamma^2}{4}\right)Q$$

$$m\ddot{Q} = -\left(k - \frac{m\gamma^2}{4}\right)Q$$

$$\boxed{\ddot{Q} + \left(\frac{k}{m} - \frac{\gamma^2}{4}\right)Q = 0}$$

Or:

$$\boxed{\ddot{Q} + \Omega^2 Q = 0}$$

where $\Omega^2 = \frac{k}{m} - \frac{\gamma^2}{4} = \omega_0^2 - \frac{\gamma^2}{4}$.

Constants of motion in new coordinates

Check the Jacobi energy:

$$\begin{aligned}
h' &= \dot{Q}\frac{\mathrm{d}L'}{\mathrm{d}\dot{Q}} - L' \\
&= \dot{Q}\left(m\dot{Q} - \frac{m\gamma}{2}Q\right) - \left[\frac{1}{2}m\dot{Q}^2 - \frac{m\gamma}{2}\dot{Q}Q - \frac{1}{2}\left(k - \frac{m\gamma^2}{4}\right)Q^2\right] \\
&= m\dot{Q}^2 - \frac{m\gamma}{2}Q\dot{Q} - \frac{1}{2}m\dot{Q}^2 + \frac{m\gamma}{2}\dot{Q}Q + \frac{1}{2}\left(k - \frac{m\gamma^2}{4}\right)Q^2 \\
&= \frac{1}{2}m\dot{Q}^2 + \frac{1}{2}\left(k - \frac{m\gamma^2}{4}\right)Q^2
\end{aligned}$$

$$\boxed{E' = \frac{1}{2}m\dot{Q}^2 + \frac{1}{2}\left(k - \frac{m\gamma^2}{4}\right)Q^2 = \text{const}}$$

Since L' has no explicit time dependence ($\frac{\mathrm{d}L'}{\mathrm{d}t} = 0$), this energy IS conserved!

Relationship between solutions

From $Q = e^{\gamma t/2}q$, we have $q = e^{-\gamma t/2}Q$.

The equation for Q is simple harmonic motion (if $\Omega^2 > 0$):

$$Q(t) = A\cos(\Omega t) + B\sin(\Omega t) = C\cos(\Omega t - \varphi)$$

where $\Omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$ (assuming underdamped).

Therefore:

$$q(t) = e^{-\gamma t/2} Q(t) = e^{-\gamma t/2} C \cos(\Omega t - \varphi)$$

This matches the solution from part (a) with $\omega_d = \Omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$!

Relationship:

The transformation $Q = e^{\gamma t/2} q$ converts the damped oscillator into an undamped one.

Original system: damped oscillator with decaying amplitude

New system: simple harmonic oscillator with conserved energy

Solutions related by: $q(t) = e^{-\gamma t/2} Q(t)$

Physical interpretation

The time-dependent Lagrangian L can be viewed as arising from a gauge transformation of the simpler Lagrangian L'. The exponential factor $e^{\gamma t}$ in L is absorbed by the coordinate transformation, revealing the underlying simple harmonic motion. The apparent "damping" in the q description is actually a consequence of the time-dependent coordinate system, not energy dissipation.

Problem 3

[30 pts] Particle on rotating hoop

A particle of mass m can move without friction on a circular hoop of radius a that rotates at constant angular velocity ω about a vertical diameter.

(a) Write down the Lagrangian and identify any constants of motion.

(b) Locate the equilibrium positions of the particle. For $\omega < \omega_c$ and $\omega > \omega_c$ where $\omega_c = \sqrt{\frac{g}{a}}$, which equilibrium positions are stable and which are unstable?

(c) For a stable equilibrium at angle θ_0, find the frequency of small oscillations about this position when $\theta(t) = \theta_0 + \varepsilon(t)$ with $|\varepsilon| \ll \theta_0$.

Solution

Part (a): Lagrangian and constants of motion

Coordinates

Use angle θ measured from the top of the hoop (vertical upward direction). The hoop rotates about the vertical diameter with constant angular velocity ω.

In a coordinate system rotating with the hoop, the particle's position is:

- Distance from axis: $\rho = a \sin\theta$
- Height below top: $z = a(1 - \cos\theta)$ (or $z = -a\cos\theta$ from center)

In the fixed (inertial) frame, if we use cylindrical coordinates (ρ, φ, z):

- $\rho = a \sin\theta$
- $\varphi = \omega t +$ (angle in rotating frame) $= \omega t$ (if we track particle's azimuthal position)
- $z = -a\cos\theta$ (taking hoop center as origin)

In the rotating frame, the particle has only one degree of freedom: θ. The azimuthal angle is fixed in the rotating frame.

Velocity in inertial frame

Position vector:

$$\vec{r} = a\sin\theta(\cos(\omega t)\hat{x} + \sin(\omega t)\hat{y}) - a\cos\theta\hat{z}$$

Velocity:

$$\vec{v} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = a\dot{\theta}\cos\theta(\cos(\omega t)\hat{x} + \sin(\omega t)\hat{y}) + a\sin\theta(-\omega\sin(\omega t)\hat{x} + \omega\cos(\omega t)\hat{y}) + a\dot{\theta}\sin\theta\hat{z}$$

Speed squared:

$$\begin{aligned}
v^2 &= a^2\dot{\theta}^2\cos^2\theta + a^2\omega^2\sin^2\theta + a^2\dot{\theta}^2\sin^2\theta \\
&= a^2\dot{\theta}^2\left(\cos^2\theta + \sin^2\theta\right) + a^2\omega^2\sin^2\theta \\
&= a^2\dot{\theta}^2 + a^2\omega^2\sin^2\theta
\end{aligned}$$

Lagrangian

Kinetic energy:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}ma^2\dot{\theta}^2 + \frac{1}{2}ma^2\omega^2\sin^2\theta$$

Potential energy (taking hoop center as reference, z positive upward):

$$V = -mga\cos\theta$$

Lagrangian:

$$\boxed{L = \frac{1}{2}ma^2\dot{\theta}^2 + \frac{1}{2}ma^2\omega^2\sin^2\theta + mga\cos\theta}$$

Constants of motion

Since ω is constant and L has no explicit t dependence, the energy (Jacobi energy) is conserved:

$$E = \dot{\theta}\frac{\mathrm{d}L}{\mathrm{d}\dot{\theta}} - L = \frac{1}{2}ma^2\dot{\theta}^2 - \frac{1}{2}ma^2\omega^2\sin^2\theta - mga\cos\theta$$

Defining the effective potential:

$$V_{\text{eff}(\theta)} = -\frac{1}{2}ma^2\omega^2\sin^2\theta - mga\cos\theta$$

$$\boxed{\textbf{Conserved quantity: } E = \tfrac{1}{2}ma^2\dot{\theta}^2 + V_{\text{eff}(\theta)} = \text{const}}$$

Part (b): Equilibrium positions

Equation of motion

From Euler-Lagrange:

$$\frac{\mathrm{d}L}{\mathrm{d}\theta} = ma^2\omega^2\sin\theta\cos\theta - mga\sin\theta$$

$$\frac{\mathrm{d}L}{\mathrm{d}\dot{\theta}} = ma^2\dot{\theta}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}L}{\mathrm{d}\dot{\theta}}\right) = ma^2\ddot{\theta}$$

Equation of motion:

$$ma^2\ddot{\theta} = ma^2\omega^2\sin\theta\cos\theta - mga\sin\theta$$

$$\ddot{\theta} = \omega^2\sin\theta\cos\theta - \frac{g}{a}\sin\theta$$

$$\boxed{\ddot{\theta} = \sin\theta\left(\omega^2\cos\theta - \frac{g}{a}\right)}$$

Equilibrium conditions

At equilibrium: $\dot{\theta} = \ddot{\theta} = 0$

From $\ddot{\theta} = \sin\theta\left(\omega^2\cos\theta - \frac{g}{a}\right) = 0$:

Either $\sin\theta = 0$ or $\omega^2\cos\theta - \frac{g}{a} = 0$.

Case 1: $\sin\theta = 0$

$$\theta = 0 \text{ (top)} \quad \text{or} \quad \theta = \pi \text{ (bottom)}$$

Case 2: $\omega^2\cos\theta = \frac{g}{a}$

$$\cos\theta = \frac{g}{a\omega^2}$$

This has real solutions only if $\left|\frac{g}{a\omega^2}\right| \leq 1$, i.e., $\omega^2 \geq \frac{g}{a}$ or $\omega \geq \omega_c$ where:

$$\omega_c = \sqrt{\frac{g}{a}}$$

For $\omega \geq \omega_c$:

$$\boxed{\theta_{\pm} = \pm \arccos\left(\frac{g}{a\omega^2}\right)}$$

(symmetric positions on either side of vertical)

Summary of equilibria:

- For all ω: $\theta = 0$ (top) and $\theta = \pi$ (bottom)
- For $\omega \geq \omega_c$: additionally $\theta = \pm \arccos\left(\frac{g}{a\omega^2}\right)$

Stability analysis

Use effective potential:

$$V_{\text{eff}(\theta)} = -\frac{1}{2} m a^2 \omega^2 \sin^2\theta - mga\cos\theta$$

Or rewrite using $\sin^2\theta = 1 - \cos^2\theta$:

$$V_{\text{eff}(\theta)} = -\frac{1}{2} m a^2 \omega^2 (1 - \cos^2\theta) - mga\cos\theta$$

$$= -\frac{1}{2} m a^2 \omega^2 + \frac{1}{2} m a^2 \omega^2 \cos^2\theta - mga\cos\theta$$

Let $u = \cos\theta$:

$$V_{\text{eff}(u)} = \text{const} + \frac{1}{2} m a^2 \omega^2 u^2 - mgau$$

Critical points:

$$\frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}u} = m a^2 \omega^2 u - mga = 0 \Longrightarrow u = \frac{g}{a\omega^2}$$

Second derivative:

$$\frac{\mathrm{d}^2 V_{\text{eff}}}{\mathrm{d}u^2} = m a^2 \omega^2 > 0$$

So $u = \frac{g}{a\omega^2}$ is a minimum of V_{eff} (stable).

At $\theta = 0$ (top):

$$\frac{\mathrm{d}^2 V_{\text{eff}}}{\mathrm{d}\theta^2}|_{\theta=0} = mga + m a^2 \omega^2 - 2 m a^2 \omega^2 = mga - m a^2 \omega^2$$

$$= ma(g - a\omega^2)$$

- If $\omega < \omega_c$ (i.e., $\omega^2 < \frac{g}{a}$): $\frac{\mathrm{d}^2 V_{\text{eff}}}{\mathrm{d}\theta^2} > 0 \to$ **stable minimum**
- If $\omega > \omega_c$ (i.e., $\omega^2 > \frac{g}{a}$): $\frac{\mathrm{d}^2 V_{\text{eff}}}{\mathrm{d}\theta^2} < 0 \to$ **unstable maximum**

At $\theta = \pi$ (bottom):

$$\frac{\mathrm{d}^2 V_{\text{eff}}}{\mathrm{d}\theta^2}|_{\theta=\pi} = mga(-1) + m a^2 \omega^2 - 2 m a^2 \omega^2$$

$$= -mga - m a^2 \omega^2 < 0$$

Always **unstable maximum**.

At $\theta = \arccos\left(\frac{g}{a\omega^2}\right)$ (for $\omega > \omega_c$):

Let $\theta_0 = \arccos\left(\frac{g}{a\omega^2}\right)$, so $\cos\theta_0 = \frac{g}{a\omega^2}$.

$$\frac{\mathrm{d}^2 V_{\text{eff}}}{\mathrm{d}\theta^2}\Big|_{\theta_0} = ma[g\cos\theta_0 + a\omega^2 - 2a\omega^2\cos^2\theta_0]$$

$$= ma\left[\frac{g^2}{a\omega^2} + a\omega^2 - \frac{2g^2}{a\omega^2}\right] = m\frac{a^2\omega^4 - g^2}{\omega^2}$$

For $\omega > \omega_c = \sqrt{\frac{g}{a}}$, we have $a^2\omega^4 > g^2$, so $\frac{\mathrm{d}^2 V_{\text{eff}}}{\mathrm{d}\theta^2} > 0 \rightarrow$ **stable minimum**.

Equilibrium stability:

For $\omega < \omega_c$:
- $\theta = 0$ (top): **stable**
- $\theta = \pi$ (bottom): **unstable**

For $\omega > \omega_c$:
- $\theta = 0$ (top): **unstable**
- $\theta = \pi$ (bottom): **unstable**
- $\theta = \pm\arccos(g/(a\omega^2))$: **stable**

Physical interpretation: Below critical speed ω_c, the particle prefers to stay at the top. Above ω_c, centrifugal effects dominate, and the particle moves to tilted equilibrium positions where gravitational and centrifugal forces balance.

Part (c): Frequency of small oscillations

For a stable equilibrium at $\theta = \theta_0$, linearize about this point.

Let $\theta(t) = \theta_0 + \varepsilon(t)$ where $|\varepsilon| \ll 1$.

From the equation of motion:

$$\ddot{\theta} = \sin\theta\left(\omega^2\cos\theta - \frac{g}{a}\right)$$

Expand to first order in ε:

$$\sin\theta = \sin(\theta_0 + \varepsilon) \approx \sin\theta_0 + \varepsilon\cos\theta_0$$

$$\cos\theta = \cos(\theta_0 + \varepsilon) \approx \cos\theta_0 - \varepsilon\sin\theta_0$$

$$\ddot{\varepsilon} \approx (\sin\theta_0 + \varepsilon\cos\theta_0)\left(\omega^2(\cos\theta_0 - \varepsilon\sin\theta_0) - \frac{g}{a}\right)$$

At equilibrium: $\sin\theta_0\left(\omega^2\cos\theta_0 - \frac{g}{a}\right) = 0$

Expanding:

$$\ddot{\varepsilon} \approx \sin\theta_0 \cdot \omega^2(\cos\theta_0 - \varepsilon\sin\theta_0) + \varepsilon\cos\theta_0\left(\omega^2\cos\theta_0 - \frac{g}{a}\right) - \sin\theta_0 \cdot \frac{g}{a}$$

Using equilibrium condition (either $\sin\theta_0 = 0$ or $\omega^2\cos\theta_0 = \frac{g}{a}$):

Case: $\boldsymbol{\theta_0 = 0}$ (top, stable for $\boldsymbol{\omega < \omega_c}$)

$\sin\theta_0 = 0$, $\cos\theta_0 = 1$:

$$\ddot{\varepsilon} \approx \varepsilon\left(\omega^2 - \frac{g}{a}\right)$$

For stability, $\omega^2 - \frac{g}{a} < 0$ (i.e., $\omega < \omega_c$):

$$\ddot{\varepsilon} + \left(\frac{g}{a} - \omega^2\right)\varepsilon = 0$$

$$\boxed{\Omega = \sqrt{\frac{g}{a} - \omega^2} = \sqrt{\omega_c^2 - \omega^2}}$$

Case: $\boldsymbol{\theta_0 = \arccos\left(\frac{g}{a\omega^2}\right)}$ (tilted, stable for $\boldsymbol{\omega > \omega_c}$)

At this equilibrium: $\omega^2 \cos\theta_0 = \frac{g}{a}$. Using the effective potential approach:

$$\ddot{\theta} = -\frac{1}{ma^2}\frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}\theta}$$

Linearizing about θ_0:

$$\ddot{\varepsilon} = -\frac{1}{ma^2}\frac{\mathrm{d}^2V_{\text{eff}}}{\mathrm{d}\theta^2}\Big|_{\theta_0}\ \varepsilon$$

From earlier: $\frac{\mathrm{d}^2V_{\text{eff}}}{\mathrm{d}\theta^2}\big|_{\theta_0} = ma^2\omega^2 - \frac{mg^2}{\omega^2}$

$$\ddot{\varepsilon} = -\left(\omega^2 - \frac{g^2}{a^2\omega^2}\right)\varepsilon$$

$$\ddot{\varepsilon} + \frac{a^2\omega^4 - g^2}{a^2\omega^2}\varepsilon = 0$$

$$\boxed{\Omega = \frac{\sqrt{a^2\omega^4 - g^2}}{a\omega} = \omega\sqrt{1 - \frac{\omega_c^4}{\omega^4}}}$$

Problem 4

[30 pts] Rotating coordinate system

A particle of mass m moves in a potential $U(r)$ (where $r = |\vec{r}|$ is the distance from origin). The system is described in spherical coordinates (r, θ, φ) that rotate at constant angular velocity Ω about the z-axis.

(a) Obtain the Lagrangian in the rotating coordinate system.

(b) Show that it can be written in the same form as in the fixed coordinate system, plus a velocity-dependent potential U' that gives rise to the centrifugal and Coriolis forces.

(c) From U', calculate the radial and azimuthal components of the centrifugal and Coriolis forces.

Solution

Part (a): Lagrangian in rotating coordinates

Coordinate systems

Fixed (inertial) frame: spherical coordinates $(r, \theta, \varphi_{\text{fixed}})$

Rotating frame: spherical coordinates $(r, \theta, \varphi_{\text{rot}})$ rotating at angular velocity Ω about z-axis

Relationship: $\varphi_{\text{fixed}} = \varphi_{\text{rot}} + \Omega t$

In the rotating frame, we use coordinates (r, θ, φ) where $\varphi = \varphi_{\text{rot}}$.

Velocity in inertial frame

Position vector in spherical coordinates:

$$\vec{r} = r\hat{r}$$

where $\hat{r}$ is the radial unit vector.

In the inertial frame, the velocity is:

$$\vec{v}_{\text{inertial}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\dot{\varphi}_{\text{fixed}}\hat{\varphi}$$

where $\hat{\theta}$ and $\hat{\varphi}$ are the standard spherical basis vectors.

Since $\varphi_{\text{fixed}} = \varphi + \Omega t$:

$$\dot{\varphi}_{\text{fixed}} = \dot{\varphi} + \Omega$$

Therefore:

$$\vec{v}_{\text{inertial}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta(\dot{\varphi} + \Omega)\hat{\varphi}$$

Kinetic energy

$$T = \frac{1}{2}mv_{\text{inertial}}^2 = \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta(\dot{\varphi} + \Omega)^2\right]$$

Expanding:

$$T = \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta(\dot{\varphi}^2 + 2\dot{\varphi}\Omega + \Omega^2)\right]$$

$$\boxed{T = \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2\right] + mr^2\sin^2\theta\Omega\dot{\varphi} + \frac{1}{2}mr^2\sin^2\theta\Omega^2}$$

Lagrangian

The potential U depends only on $r = |\vec{r}|$, so:

$$\boxed{L = \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2\right] + mr^2\sin^2\theta\Omega\dot{\varphi} + \frac{1}{2}mr^2\sin^2\theta\Omega^2 - U(r)}$$

Part (b): Velocity-dependent potential

The Lagrangian in a fixed (non-rotating) frame would be:

$$L_{\text{fixed}} = \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2\right] - U(r)$$

Comparing with the rotating frame Lagrangian:

$$L = L_{\text{fixed}} + mr^2\sin^2\theta\Omega\dot{\varphi} + \frac{1}{2}mr^2\sin^2\theta\Omega^2$$

The additional terms can be written as:

$$U' = -mr^2\sin^2\theta\Omega\dot{\varphi} - \frac{1}{2}mr^2\sin^2\theta\Omega^2$$

So:

$$\boxed{L = L_{\text{fixed}} - U'(r, \theta, \dot{\varphi})}$$

where:

$$\boxed{U' = mr^2\sin^2\theta\Omega\dot{\varphi} + \frac{1}{2}mr^2\sin^2\theta\Omega^2}$$

This is a **velocity-dependent potential** (depends on $\dot{\varphi}$).

Generalized forces from U'

The generalized force is:

$$Q_i = -\frac{\mathrm{d}U'}{\mathrm{d}q_i} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}U'}{\mathrm{d}\dot{q}_i}\right)$$

For r:

$$\frac{\mathrm{d}U'}{\mathrm{d}r} = 2mr\sin^2\theta\Omega\dot{\varphi} + mr\sin^2\theta\Omega^2$$

$$\frac{\mathrm{d}U'}{\mathrm{d}\dot{r}} = 0 \Longrightarrow \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}U'}{\mathrm{d}\dot{r}}\right) = 0$$

$$Q_r = -2mr\sin^2\theta\Omega\dot{\varphi} - mr\sin^2\theta\Omega^2$$

For θ:

$$\frac{\mathrm{d}U'}{\mathrm{d}\theta} = 2mr^2\sin\theta\cos\theta\Omega\dot{\varphi} + mr^2\sin\theta\cos\theta\Omega^2$$

$$Q_\theta = -2mr^2\sin\theta\cos\theta\Omega\dot{\varphi} - mr^2\sin\theta\cos\theta\Omega^2$$

For φ:

$$\frac{\mathrm{d}U'}{\mathrm{d}\varphi} = 0$$

$$\frac{\mathrm{d}U'}{\mathrm{d}\dot{\varphi}} = mr^2 \sin^2\theta\Omega$$

$$Q_\varphi = -0 + \frac{\mathrm{d}}{\mathrm{d}t}(mr^2 \sin^2\theta\Omega) = m\Omega\frac{\mathrm{d}}{\mathrm{d}t}(r^2 \sin^2\theta)$$

$$= m\Omega\left(2r\dot{r}\sin^2\theta + r^2 \cdot 2\sin\theta\cos\theta\dot{\theta}\right)$$

$$= 2m\Omega\sin\theta\left(r\dot{r}\sin\theta + r^2\cos\theta\dot{\theta}\right)$$

Part (c): Centrifugal and Coriolis forces

The pseudo-forces in the rotating frame are the centrifugal force and Coriolis force.

From the velocity-dependent potential U', we can identify:

Centrifugal potential:

$$U_{\text{centrifugal}} = -\frac{1}{2}mr^2\sin^2\theta\Omega^2 = -\frac{1}{2}m\Omega^2\rho^2$$

where $\rho = r\sin\theta$ is the perpendicular distance from the rotation axis.

The centrifugal force is:

$$\vec{F}_{\text{centrifugal}} = -\nabla U_{\text{centrifugal}} = m\Omega^2\rho\hat{\rho}$$

(outward, perpendicular to rotation axis)

Coriolis potential:

$$U_{\text{Coriolis}} = -mr^2\sin^2\theta\Omega\dot{\varphi}$$

This is velocity-dependent, giving the Coriolis force.

Components of forces

In spherical coordinates, the position-dependent parts of Q_r, Q_θ give:

From $Q_r = -2mr\sin^2\theta\Omega\dot{\varphi} - mr\sin^2\theta\Omega^2$:

- Centrifugal (from $-mr\sin^2\theta\Omega^2$):

$$\boxed{F_r^{(\text{centrifugal})} = mr\sin^2\theta\Omega^2 = m\Omega^2\frac{\rho^2}{r}}$$

  (outward radial component)

- Coriolis (from $-2mr\sin^2\theta\Omega\dot{\varphi}$):

$$\boxed{F_r^{(\text{Coriolis})} = 2mr\sin^2\theta\Omega\dot{\varphi}}$$

From $Q_\theta = -2mr^2\sin\theta\cos\theta\Omega\dot{\varphi} - mr^2\sin\theta\cos\theta\Omega^2$:

- Centrifugal:

$$\boxed{F_\theta^{(\text{centrifugal})} = mr^2\sin\theta\cos\theta\Omega^2 = mr\sin\theta\cos\theta\Omega^2 \cdot r}$$

- Coriolis:

$$\boxed{F_\theta^{(\text{Coriolis})} = 2mr^2 \sin\theta \cos\theta \Omega\dot{\varphi}}$$

From Q_φ:

The azimuthal force comes from:

$$Q_\varphi = 2m\Omega \sin\theta \left(r\dot{r}\sin\theta + r^2\cos\theta\dot{\theta}\right)$$

This is purely Coriolis (velocity-dependent):

$$\boxed{F_\varphi^{(\text{Coriolis})} = 2m\Omega \sin\theta \left(r\dot{r}\sin\theta + r^2\cos\theta\dot{\theta}\right)}$$

(no centrifugal component in azimuthal direction, as centrifugal force is radial)

Summary in component form

Centrifugal force:

$$F_r^{(\text{cent})} = m\Omega^2 r \sin^2\theta$$

$$F_\theta^{(\text{cent})} = m\Omega^2 r \sin\theta\cos\theta$$

$$F_\varphi^{(\text{cent})} = 0$$

Coriolis force:

$$F_r^{(\text{Cor})} = 2m\Omega r \sin^2\theta\dot{\varphi}$$

$$F_\theta^{(\text{Cor})} = 2m\Omega r^2 \sin\theta\cos\theta\dot{\varphi}$$

$$F_\varphi^{(\text{Cor})} = 2m\Omega r \sin\theta\left(\dot{r}\sin\theta + r\dot{\theta}\cos\theta\right)$$

Physical interpretation

- **Centrifugal force**: Points outward from the rotation axis, with magnitude $m\Omega^2\rho$ where $\rho = r\sin\theta$ is the perpendicular distance from axis. Components:
  - Radial: $m\Omega^2 r\sin^2\theta$ (outward)
  - Polar: $m\Omega^2 r\sin\theta\cos\theta$ (toward equator if in northern hemisphere)
- **Coriolis force**: Perpendicular to velocity in rotating frame, given by $\vec{F}_{\text{Cor}} = -2m\vec{\Omega}\times\vec{v}_{\text{rot}}$. Components depend on velocities $\dot{r}, \dot{\theta}, \dot{\varphi}$.

Vector form

In Cartesian coordinates, these are:

$$\vec{F}_{\text{centrifugal}} = m\Omega^2\rho\hat{\rho} = m\vec{\Omega}\times\left(\vec{\Omega}\times\vec{r}\right)$$

$$\vec{F}_{\text{Coriolis}} = -2m\vec{\Omega}\times\vec{v}_{\text{rot}}$$

where $\vec{\Omega} = \Omega\hat{z}$ and $\vec{v}_{\text{rot}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\dot{\varphi}\hat{\varphi}$.