Assignment 3

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PC3261
Classical Mechanics II

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Problem 1

[20 pts] Sphere on incline

A uniform sphere has mass M and radius a. The sphere is released from rest on a plane that is inclined at an angle β to the horizontal and for which the coefficient of static friction is μ_s.

(a) Suppose the sphere rolls without slipping. Determine the acceleration of the CM and the required frictional force in terms of M, g and β. Deduce that there is a critical angle of inclination β_c below which there is pure rolling motion and above which there is slipping.

(b) Suppose $\beta > \beta_c$ so that that slipping occurs. The coefficient of kinetic friction between the sphere and the plane is μ_k. Determine the acceleration of the CM and the angular acceleration. Deduce a relation between the CM velocity and the angular velocity.

(c) Show that the mechanical energy of the sphere is constant during rolling but decreases quadratically with time during slipping.

Solution

Part (a): Rolling without slipping

Setup and force analysis

For a uniform sphere, the moment of inertia about the center is:

$$I = \frac{2}{5}Ma^2$$

Let x be the position of the center of mass down the incline, with acceleration $a_{\text{CM}} = \ddot{x}$.

Forces on the sphere:

- Weight component along incline: $Mg\sin\beta$ (down the incline)
- Normal force: $N = Mg\cos\beta$ (perpendicular to incline)
- Friction force: f (up the incline, opposing motion)

Newton's second law for translation:

$$Ma_{\text{CM}} = Mg\sin\beta - f \qquad (1)$$

Rotational equation

The friction force creates a torque about the center of mass:

$$\tau = fa = I\alpha = \frac{2}{5}Ma^2\alpha \qquad (2)$$

where $\alpha = \ddot{\theta}$ is the angular acceleration.

Rolling constraint

For rolling without slipping:

$$x = a\theta \Longrightarrow a_{\text{CM}} = a\alpha \qquad (3)$$

From equation (2):

$$f = \frac{I\alpha}{a} = \frac{\frac{2}{5}Ma^2\alpha}{a} = \frac{2}{5}Ma\alpha = \frac{2}{5}Ma_{\text{CM}}$$

Substituting into equation (1):

$$Ma_{\text{CM}} = Mg\sin\beta - \frac{2}{5}Ma_{\text{CM}}$$

$$Ma_{\text{CM}} + \frac{2}{5}Ma_{\text{CM}} = Mg\sin\beta$$

$$\frac{7}{5}Ma_{\text{CM}} = Mg\sin\beta$$

$$\boxed{a_{\text{CM}} = \frac{5}{7}g\sin\beta}$$

The frictional force is:

$$\boxed{f = \frac{2}{5}Ma_{\text{CM}} = \frac{2}{5}M \cdot \frac{5}{7}g\sin\beta = \frac{2}{7}Mg\sin\beta}$$

Critical angle

Slipping begins when the friction force reaches its maximum value:

$$f_{\max} = \mu_s N = \mu_s Mg\cos\beta$$

At the critical angle β_c:

$$\frac{2}{7}Mg\sin\beta_c = \mu_s Mg\cos\beta_c$$

$$\tan\beta_c = \frac{7}{2}\mu_s$$

$$\boxed{\beta_c = \arctan\left(\frac{7}{2}\mu_s\right)}$$

Part (b): Slipping motion

When $\beta > \beta_c$, the sphere slips. The kinetic friction force is:

$$f = \mu_k N = \mu_k Mg\cos\beta$$

Linear acceleration

From Newton's second law:

$$Ma_{\text{CM}} = Mg\sin\beta - \mu_k Mg\cos\beta$$

$$\boxed{a_{\text{CM}} = g(\sin\beta - \mu_k\cos\beta)}$$

Angular acceleration

The torque equation:

$$I\alpha = fa = \mu_k Mg\cos\beta \cdot a$$

$$\frac{2}{5}Ma^2\alpha = \mu_k Mga\cos\beta$$

$$\boxed{\alpha = \frac{5\mu_k g\cos\beta}{2a}}$$

Relationship between v_{CM} and ω

Since slipping occurs, the rolling constraint is violated. Integrating the accelerations from rest:

$$v_{\mathrm{CM}} = a_{\mathrm{CM}}t = g(\sin\beta - \mu_k\cos\beta)t$$

$$\omega = \alpha t = \frac{5\mu_k g\cos\beta}{2a}t$$

Eliminating t:

$$\omega = \frac{5\mu_k g\cos\beta}{2a} \cdot \frac{v_{\mathrm{CM}}}{g(\sin\beta - \mu_k\cos\beta)}$$

$$\boxed{\omega = \frac{5\mu_k\cos\beta}{2a(\sin\beta - \mu_k\cos\beta)}v_{\mathrm{CM}}}$$

Alternatively, expressing v_{CM} in terms of ω:

$$\boxed{v_{\mathrm{CM}} = \frac{2a(\sin\beta - \mu_k\cos\beta)}{5\mu_k\cos\beta}\omega}$$

Note that during slipping, $v_{\mathrm{CM}} > a\omega$ (the contact point slides down the incline).

Part (c): Energy analysis

Case 1: Rolling without slipping

The total mechanical energy is:

$$E = \frac{1}{2}Mv_{\mathrm{CM}}^2 + \frac{1}{2}I\omega^2 + Mgh = \frac{7}{10}Mv_{\mathrm{CM}}^2 + Mgh$$

where we used $I = \frac{2}{5}Ma^2$ and the rolling constraint $v_{\mathrm{CM}} = a\omega$.

Taking the time derivative:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{7}{5}Mv_{\mathrm{CM}}a_{\mathrm{CM}} + Mg\dot{h}$$

Since $\dot{h} = -v_{\mathrm{CM}}\sin\beta$ and $a_{\mathrm{CM}} = \frac{5}{7}g\sin\beta$:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{7}{5}Mv_{\mathrm{CM}} \cdot \frac{5}{7}g\sin\beta - Mgv_{\mathrm{CM}}\sin\beta = 0$$

$$\boxed{\frac{\mathrm{d}E}{\mathrm{d}t} = 0 \quad \text{during rolling without slipping}}$$

Energy is conserved because the friction force does no work (the contact point has zero velocity).

Case 2: Slipping motion

The total energy is:

$$E = \frac{1}{2}Mv_{\text{CM}}^2 + \frac{1}{5}Ma^2\omega^2 + Mgh$$

The rate of energy dissipation is the power lost to friction:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -f \cdot v_{\text{slip}} = -\mu_k Mg\cos\beta(v_{\text{CM}} - a\omega)$$

where $v_{\text{slip}} = v_{\text{CM}} - a\omega$ is the slip velocity at the contact point.

Quadratic decrease with time

From part (b), during slipping (starting from rest):

$$v_{\text{CM}} = g(\sin\beta - \mu_k\cos\beta)t, \quad \omega = \frac{5\mu_k g\cos\beta}{2a}t$$

The height decreases as: $h(t) = h_0 - \frac{1}{2}g(\sin\beta - \mu_k\cos\beta)\sin\beta t^2$

Substituting into the energy expression:

$$E(t) = \frac{1}{2}M[g(\sin\beta - \mu_k\cos\beta)t]^2 + \frac{1}{5}Ma^2\left[\frac{5\mu_k g\cos\beta}{2a}t\right]^2 + Mg\left[h_0 - \frac{1}{2}g(\sin\beta - \mu_k\cos\beta)\sin\beta t^2\right]$$

$$= Mgh_0 + Mg^2t^2\left[\frac{1}{2}(\sin\beta - \mu_k\cos\beta)^2 + \left(\frac{5}{4}\right)\mu_k^2\cos^2\beta - \frac{1}{2}(\sin\beta - \mu_k\cos\beta)\sin\beta\right]$$

Simplifying the bracket:

$$[\cdot] = \frac{1}{2}(\sin^2\beta - 2\mu_k\sin\beta\cos\beta + \mu_k^2\cos^2\beta) + \left(\frac{5}{4}\right)\mu_k^2\cos^2\beta - \frac{1}{2}\sin^2\beta + \frac{1}{2}\mu_k\sin\beta\cos\beta$$

$$= -\frac{1}{2}\mu_k\sin\beta\cos\beta + \left(\frac{7}{4}\right)\mu_k^2\cos^2\beta$$

Therefore:

$$E(t) = Mgh_0 - Mg^2t^2\left[\frac{1}{2}\mu_k\sin\beta\cos\beta - \left(\frac{7}{4}\right)\mu_k^2\cos^2\beta\right]$$

The coefficient of t^2 is positive (since $\mu_k < \tan\beta$ for slipping), so:

$$\boxed{E(t) = E_0 - Ct^2 \quad \text{where } C > 0}$$

where $E_0 = Mgh_0$ and $C = Mg^2\left[\frac{1}{2}\mu_k\sin\beta\cos\beta - \left(\frac{7}{4}\right)\mu_k^2\cos^2\beta\right]$.

$$\boxed{\begin{array}{c}\text{Energy decreases quadratically with time during slipping,}\\ \text{whereas it remains constant during pure rolling}\end{array}}$$

Problem 2

[20 pts] Rope sliding off table

A uniform rope of length ℓ and mass M is stretched on a table with a segment of length x_0 hanging over the edge. The rope is released from rest.

(a) Suppose the table is frictionless, find the time t_s taken for the rope to slide off the table from Newton's second law.

(b) Solve the same problem by applying work-energy theorem, and compare the result with (a).

(c) Suppose the motion is subjected to friction, with a coefficient of kinetic friction μ_k between the rope and table. Find the time t_s taken for the rope to slide off the table.

(d) Calculate the total loss in mechanical energy of the rope at time t_s in terms of μ_k, $\frac{x_0}{\ell}$ and $Mg\ell$.

Solution

Part (a): Frictionless case using Newton's second law

Setup

Let $x(t)$ be the length of rope hanging over the edge at time t. Initially, $x(0) = x_0$ and $\dot{x}(0) = 0$.

The linear mass density is $\lambda = \frac{M}{\ell}$.

Force analysis

The hanging segment has mass $m_{\text{hang}} = \lambda x = \frac{Mx}{\ell}$ and experiences gravitational force $m_{\text{hang}}g = \frac{Mgx}{\ell}$ downward.

The portion on the table has mass $m_{\text{table}} = \frac{M(\ell-x)}{\ell}$ and experiences no net horizontal force (frictionless table, no tension gradient).

Newton's second law

The entire rope moves with the same acceleration $a = \ddot{x}$. Applying Newton's second law to the whole rope:

$$M\ddot{x} = \frac{Mgx}{\ell}$$
$$\ddot{x} = \frac{gx}{\ell}$$

Solving the differential equation

Rearranging:

$$\ddot{x} - \left(\frac{g}{\ell}\right)x = 0$$

The general solution is:

$$x(t) = Ae^{\sqrt{\frac{g}{\ell}}t} + Be^{-\sqrt{\frac{g}{\ell}}t}$$

Applying initial conditions $x(0) = x_0$ and $\dot{x}(0) = 0$:

$$x(0) = A + B = x_0$$
$$\dot{x}(t) = A\sqrt{\frac{g}{\ell}}e^{\sqrt{\frac{g}{\ell}}t} - B\sqrt{\frac{g}{\ell}}e^{-\sqrt{\frac{g}{\ell}}t}$$
$$\dot{x}(0) = \sqrt{\frac{g}{\ell}}(A - B) = 0 \Longrightarrow A = B$$

From $A + B = x_0$ and $A = B$:

$$A = B = \frac{x_0}{2}$$

Therefore:

$$x(t) = \frac{x_0}{2}\left[e^{\sqrt{\frac{g}{\ell}}t} + e^{-\sqrt{\frac{g}{\ell}}t}\right] = x_0 \cosh\left(\sqrt{\frac{g}{\ell}}t\right)$$

Finding the time to slide off

The rope completely slides off when $x(t_s) = \ell$:

$$\ell = x_0 \cosh\left(\sqrt{\frac{g}{\ell}}t_s\right)$$

$$\cosh\left(\sqrt{\frac{g}{\ell}}t_s\right) = \frac{\ell}{x_0}$$

$$\sqrt{\frac{g}{\ell}}t_s = \operatorname{arccosh}\left(\frac{\ell}{x_0}\right) = \ln\left(\frac{\ell}{x_0} + \sqrt{\left(\frac{\ell}{x_0}\right)^2 - 1}\right)$$

$$\boxed{t_s = \sqrt{\frac{\ell}{g}}\ln\left(\frac{\ell}{x_0} + \sqrt{\left(\frac{\ell}{x_0}\right)^2 - 1}\right)}$$

Alternatively:

$$\boxed{t_s = \sqrt{\frac{\ell}{g}}\ \operatorname{arccosh}\left(\frac{\ell}{x_0}\right)}$$

Part (b): Work-energy theorem

Energy method

The initial potential energy (taking the table as reference level $U = 0$):

$$E_0 = -\int_0^{x_0} \lambda g y\,\mathrm{d}y = -\lambda g \cdot \frac{x_0^2}{2} = -\frac{Mgx_0^2}{2\ell}$$

(negative because the hanging segment is below the table)

When length x hangs over the edge, the potential energy is:

$$U(x) = -\int_0^{x} \lambda g y\,\mathrm{d}y = -\frac{Mgx^2}{2\ell}$$

The kinetic energy is:

$$K(x) = \frac{1}{2}M\dot{x}^2$$

By energy conservation:

$$K(x) + U(x) = E_0$$

$$\frac{1}{2}M\dot{x}^2 - \frac{Mgx^2}{2\ell} = -\frac{Mgx_0^2}{2\ell}$$

$$\dot{x}^2 = \frac{g(x^2 - x_0^2)}{\ell}$$

$$\dot{x} = \sqrt{\frac{g(x^2 - x_0^2)}{\ell}}$$

(taking positive root since x increases with time)

Separating variables

$$\mathrm{d}t = \frac{\mathrm{d}x}{\sqrt{\frac{g(x^2-x_0^2)}{\ell}}} = \sqrt{\frac{\ell}{g}}\frac{\mathrm{d}x}{\sqrt{x^2 - x_0^2}}$$

Integrating from x_0 to ℓ:

$$t_s = \sqrt{\frac{\ell}{g}}\int_{x_0}^{\ell}\frac{\mathrm{d}x}{\sqrt{x^2 - x_0^2}}$$

Using the standard integral $\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}} = \mathrm{arccosh}\left(\frac{x}{a}\right) + C$:

$$t_s = \sqrt{\frac{\ell}{g}}\left[\mathrm{arccosh}\left(\frac{x}{x_0}\right)\right]_{x_0}^{\ell}$$

$$= \sqrt{\frac{\ell}{g}}\left[\mathrm{arccosh}\left(\frac{\ell}{x_0}\right) - \mathrm{arccosh}(1)\right]$$

$$= \sqrt{\frac{\ell}{g}}\ \mathrm{arccosh}\left(\frac{\ell}{x_0}\right)$$

(since $\mathrm{arccosh}(1) = 0$)

$$\boxed{t_s = \sqrt{\frac{\ell}{g}}\ \mathrm{arccosh}\left(\frac{\ell}{x_0}\right) = \sqrt{\frac{\ell}{g}}\ln\left(\frac{\ell}{x_0} + \sqrt{\left(\frac{\ell}{x_0}\right)^2 - 1}\right)}$$

This result agrees exactly with part (a), confirming the consistency of the two methods.

Part (c): With kinetic friction

Force analysis with friction

The friction force on the portion of rope on the table is:

$$f = \mu_k N = \mu_k m_{\text{table}} g = \mu_k Mg\frac{\ell - x}{\ell}$$

This opposes the motion (acts backward on the table portion).

Newton's second law for the entire rope:

$$M\ddot{x} = \frac{Mgx}{\ell} - \mu_k Mg\frac{\ell - x}{\ell}$$

$$\ddot{x} = \frac{gx}{\ell} - \frac{\mu_k g(\ell - x)}{\ell}$$

$$\ddot{x} = \frac{g}{\ell}[x - \mu_k(\ell - x)] = \frac{g}{\ell}[x + \mu_k x - \mu_k \cdot \ell]$$

$$\ddot{x} = \frac{g}{\ell}[(1 + \mu_k)x - \mu_k \cdot \ell]$$

Solving the differential equation

Rearranging:

$$\ddot{x} - \frac{g(1 + \mu_k)}{\ell}x = -(\mu_k g)$$

This is a second-order linear ODE with constant coefficients. Let $\omega^2 = \frac{g(1+\mu_k)}{\ell}$.

The homogeneous solution is:

$$x_{h(t)} = Ae^{\omega t} + Be^{-\omega t}$$

For the particular solution, try a constant: $x_p = C$:

$$0 - \omega^2 C = -\mu_k \cdot g$$

$$C = \frac{\mu_k g}{\omega^2} = \frac{\mu_k g}{\frac{g(1+\mu_k)}{\ell}} = \frac{\mu_k \ell}{1 + \mu_k}$$

General solution:

$$x(t) = Ae^{\omega t} + Be^{-\omega t} + \frac{\mu_k \ell}{1 + \mu_k}$$

Applying initial conditions

At $t = 0$: $x(0) = x_0$, $\dot{x}(0) = 0$:

$$x_0 = A + B + \frac{\mu_k \ell}{1 + \mu_k}$$

$$A + B = x_0 - \frac{\mu_k \cdot \ell}{1 + \mu_k} = \frac{x_0(1 + \mu_k) - \mu_k \cdot \ell}{1 + \mu_k}$$

$$\dot{x}(t) = A\omega e^{\omega t} - B\omega e^{-\omega t}$$

$$\dot{x}(0) = \omega(A - B) = 0 \Longrightarrow A = B$$

Therefore:

$$A = B = \frac{1}{2} \cdot \frac{x_0(1 + \mu_k) - \mu_k \cdot \ell}{1 + \mu_k} = \frac{x_0(1 + \mu_k) - \mu_k \cdot \ell}{2(1 + \mu_k)}$$

The solution is:

$$\begin{aligned} x(t) &= \frac{x_0(1 + \mu_k) - \mu_k \cdot \ell}{2(1 + \mu_k)}[e^{\omega t} + e^{-\omega t}] + \frac{\mu_k \cdot \ell}{1 + \mu_k} \\ &= \frac{x_0(1 + \mu_k) - \mu_k \cdot \ell}{1 + \mu_k}\cosh(\omega t) + \frac{\mu_k \cdot \ell}{1 + \mu_k} \\ &= \frac{(x_0(1 + \mu_k) - \mu_k \cdot \ell)\cosh(\omega t) + \mu_k \cdot \ell}{1 + \mu_k} \end{aligned}$$

Finding time to slide off

When $x(t_s) = \ell$:

$$\ell = \frac{(x_0(1 + \mu_k) - \mu_k \cdot \ell)\cosh(\omega t_s) + \mu_k \cdot \ell}{1 + \mu_k}$$

$$\ell(1 + \mu_k) = (x_0(1 + \mu_k) - \mu_k \cdot \ell)\cosh(\omega t_s) + \mu_k \cdot \ell$$

$$\ell(1+\mu_k) - \mu_k \cdot \ell = (x_0(1+\mu_k) - \mu_k \cdot \ell)\cosh(\omega t_s)$$
$$\ell = (x_0(1+\mu_k) - \mu_k \cdot \ell)\cosh(\omega t_s)$$
$$\cosh(\omega t_s) = \frac{\ell}{x_0(1+\mu_k) - \mu_k \cdot \ell} = \frac{\ell}{x_0 + \mu_k \cdot (x_0 - \ell)}$$

With $\omega = \sqrt{\frac{g(1+\mu_k)}{\ell}}$:

$$\boxed{t_s = \sqrt{\frac{\ell}{g(1+\mu_k)}}\,\operatorname{arccosh}\left(\frac{\ell}{x_0 + \mu_k \cdot (x_0 - \ell)}\right)}$$

Alternatively:

$$\boxed{t_s = \sqrt{\frac{\ell}{g(1+\mu_k)}}\ln\left(\frac{\ell + \sqrt{\ell^2 - [x_0 + \mu_k \cdot (x_0 - \ell)]^2}}{x_0 + \mu_k \cdot (x_0 - \ell)}\right)}$$

Note: For $\mu_k = 0$, this reduces to the frictionless case.

Part (d): Energy loss

Initial and final energies

Initial energy (at $t = 0$, $x = x_0$):

$$E_0 = -\frac{Mgx_0^2}{2\ell}$$

Final energy (at $t = t_s$, $x = \ell$):

$$E_f = \frac{1}{2}Mv_f^2 - \frac{Mg\ell^2}{2\ell} = \frac{1}{2}Mv_f^2 - \frac{Mg\ell}{2}$$

where $v_f = \dot{x}(t_s)$.

From $x(t) = \frac{(x_0(1+\mu_k) - \mu_k \cdot \ell)\cosh(\omega t) + \mu_k \cdot \ell}{1+\mu_k}$:

$$\dot{x}(t) = \frac{(x_0(1+\mu_k) - \mu_k \cdot \ell)\omega \sinh(\omega t)}{1+\mu_k}$$

At $t = t_s$, using $\cosh(\omega t_s) = \frac{\ell}{x_0 + \mu_k \cdot (x_0 - \ell)}$:

From $\cosh^2 - \sinh^2 = 1$:

$$\begin{aligned}
\sinh^2(\omega t_s) &= \cosh^2(\omega t_s) - 1 \\
&= \left[\frac{\ell}{x_0 + \mu_k \cdot (x_0 - \ell)}\right]^2 - 1 \\
&= \frac{\ell^2 - [x_0 + \mu_k \cdot (x_0 - \ell)]^2}{[x_0 + \mu_k \cdot (x_0 - \ell)]^2}
\end{aligned}$$

Let $x_{\text{eff}} = x_0 + \mu_k \cdot (x_0 - \ell) = x_0(1+\mu_k) - \mu_k \cdot \ell$:

$$\sinh(\omega t_s) = \frac{\sqrt{\ell^2 - x_{\text{eff}}^2}}{x_{\text{eff}}}$$

Therefore:

$$
\begin{aligned}
v_f &= \frac{x_{\text{eff}} \cdot \omega \cdot \sqrt{\ell^2 - x_{\text{eff}}^2}}{x_{\text{eff}}(1+\mu_k)} \\
&= \frac{\omega\sqrt{\ell^2 - x_{\text{eff}}^2}}{1+\mu_k} \\
&= \frac{\sqrt{\frac{g(1+\mu_k)}{\ell}}\sqrt{\ell^2 - x_{\text{eff}}^2}}{1+\mu_k} \\
&= \sqrt{\frac{g}{\ell}} \cdot \frac{\sqrt{\ell^2 - x_{\text{eff}}^2}}{\sqrt{1+\mu_k}} \\
v_f^2 &= \frac{g\left(\ell^2 - x_{\text{eff}}^2\right)}{\ell(1+\mu_k)}
\end{aligned}
$$

Energy loss calculation

The energy loss is:

$$
\begin{aligned}
\Delta E = E_f - E_0 &= \left[\frac{1}{2}Mv_f^2 - \frac{Mg\ell}{2}\right] - \left[-\frac{Mgx_0^2}{2\ell}\right] \\
&= \frac{1}{2}Mv_f^2 - \frac{Mg\ell}{2} + \frac{Mgx_0^2}{2\ell}
\end{aligned}
$$

Substituting v_f^2:

$$
\begin{aligned}
\Delta E &= \frac{1}{2}M \cdot \frac{g\left(\ell^2 - x_{\text{eff}}^2\right)}{\ell(1+\mu_k)} - \frac{Mg\ell}{2} + \frac{Mgx_0^2}{2\ell} \\
&= \frac{Mg}{2\ell}\left[\frac{\ell^2 - x_{\text{eff}}^2}{1+\mu_k} - \ell^2 + x_0^2\right]
\end{aligned}
$$

With $x_{\text{eff}} = x_0(1+\mu_k) - \mu_k \cdot \ell$:

$$
x_{\text{eff}}^2 = [x_0(1+\mu_k) - \mu_k \cdot \ell]^2 = x_0^2(1+\mu_k)^2 - 2x_0\mu_k \cdot \ell(1+\mu_k) + \mu_k^2\ell^2
$$

Energy dissipated by friction

The work done against friction as the rope slides off is:

$$
W_{\text{friction}} = \int_0^{t_s} f \cdot v \, \mathrm{d}t = \int_0^{t_s} \mu_k \cdot Mg\frac{\ell - x}{\ell} \cdot \dot{x} \, \mathrm{d}t
$$

Changing variables to x:

$$
\begin{aligned}
W_{\text{friction}} &= \int_{x_0}^{\ell} \mu_k \cdot Mg\frac{\ell - x}{\ell} \, \mathrm{d}x \\
&= \frac{\mu_k Mg}{\ell}\int_{x_0}^{\ell} (\ell - x) \, \mathrm{d}x \\
&= \frac{\mu_k Mg}{\ell}\left[\ell x - \frac{x^2}{2}\right]_{x_0}^{\ell} \\
&= \frac{\mu_k Mg}{\ell}\left[\left(\ell^2 - \frac{\ell^2}{2}\right) - \left(\ell x_0 - \frac{x_0^2}{2}\right)\right]
\end{aligned}
$$

$$= \frac{\mu_k M g}{\ell}\left[\frac{\ell^2}{2} - \ell x_0 + \frac{x_0^2}{2}\right]$$

$$= \frac{\mu_k M g}{2\ell}[\ell^2 - 2\ell x_0 + x_0^2]$$

$$= \frac{\mu_k M g}{2\ell}(\ell - x_0)^2$$

The total energy loss (magnitude) is:

$$\boxed{|\Delta E| = \frac{\mu_k M g}{2\ell}(\ell - x_0)^2 = \frac{\mu_k M g \ell}{2}\left(1 - \frac{x_0}{\ell}\right)^2}$$

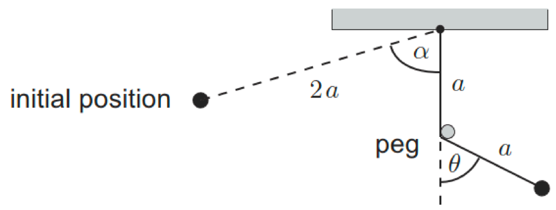
Expressing in the requested form:

$$\boxed{|\Delta E| = \frac{1}{2}\mu_k \cdot M g \ell\left(1 - \frac{x_0}{\ell}\right)^2}$$

The energy loss is proportional to μ_k and to the square of the distance $(\ell - x_0)$ that slides across the table. Longer sliding distances and higher friction lead to greater dissipation.

Problem 3

[20 pts] Ball and peg



A heavy ball is attached to a fixed point O by a light inextensible string of length $2a$. The ball is drawn back until the string makes an acute angle α with the vertical and is then released. A thin peg is fixed a distance a vertically below O in the path of the string as shown.

In a game, the contestant chooses the value of α and wins a prize if the ball strikes the peg. Ignoring frictions, find the winning value of α.

Solution

Setup

The ball is released from angle α from vertical. The string wraps around the peg when the ball passes through the vertical position on the opposite side. After wrapping, the ball swings in a circle of radius a centered at the peg.

Phase 1: Initial swing

Taking the peg as reference level ($h = 0$ at peg), the initial height is:

$$h_i = a - 2a\cos\alpha = a(1 - 2\cos\alpha)$$

where a is the distance from O to the peg.

The lowest point of the swing (ball directly below O) is at height:

$$h_{\text{bottom}} = a - 2a = -a$$

By energy conservation:

$$mgh_i = mgh_{\text{bottom}} + \frac{1}{2}mv_{\text{bottom}}^2$$

$$ga(1 - 2\cos\alpha) = -ga + \frac{1}{2}v_{\text{bottom}}^2$$

$$v_{\text{bottom}}^2 = 2ga(1 - 2\cos\alpha + 1) = 4ga(1 - \cos\alpha)$$

Phase 2: After wrapping around peg

When the string wraps around the peg, the effective radius becomes a. The ball is at the bottom of the new circular path (height $-a$ relative to peg) with speed v_{bottom}.

For the ball to strike the peg, it must reach the peg height ($h = 0$) with zero velocity, at which point the string goes slack and the ball falls onto the peg.

Energy conservation from bottom to peg level

At the bottom of the circle (height $-a$), the ball has speed v_{bottom}.

At the peg height ($h = 0$), the ball should have zero speed:

$$\frac{1}{2}mv_{\text{bottom}}^2 - mga = mg \cdot 0 + \frac{1}{2}m \cdot 0^2$$

$$\frac{1}{2}v_{\text{bottom}}^2 = ga$$

$$v_{\text{bottom}}^2 = 2ga$$

From $v_{\text{bottom}}^2 = 4ga(1 - \cos\alpha)$:

$$4ga(1 - \cos\alpha) = 2ga$$

$$1 - \cos\alpha = \frac{1}{2}$$

$$\cos\alpha = \frac{1}{2}$$

$$\boxed{\alpha = 60^\circ = \frac{\pi}{3}}$$

Problem 4

[20 pts] Elastic cord

A light elastic cord of length 2ℓ and spring constant k is held with the ends fixed a distance 2ℓ apart in a horizontal position. A block of mass m is then suspended from the midpoint of the cord.

(a) Choosing zero potential reference at $y = 0$ before the block is suspended from the mid point of the cord. Find an expression for the potential energy of the system $V(y)$ where y is the vertical sag of the center of the cord.

(b) Determine the frequency of vertical oscillations about the equilibrium position of the block.

Solution

Part (a): Potential energy

Geometry

Initially, the cord is horizontal with endpoints separated by 2ℓ. The midpoint is at the origin ($y = 0$).

When the block is suspended and the midpoint sags by distance y (downward, $y > 0$), each half of the cord forms a straight line from an endpoint to the midpoint.

Taking y as positive downward: endpoints at $(\pm\ell, 0)$, midpoint at $(0, y)$.

Length of each half

The length of each half is:

$$L = \sqrt{\ell^2 + y^2}$$

The total length of the cord is:

$$L_{\text{total}} = 2\sqrt{\ell^2 + y^2}$$

The natural length is 2ℓ, so the extension is:

$$\Delta L = 2\sqrt{\ell^2 + y^2} - 2\ell = 2\left(\sqrt{\ell^2 + y^2} - \ell\right)$$

Elastic potential energy

For a spring with spring constant k and extension ΔL:

$$U_{\text{elastic}} = \frac{1}{2}k(\Delta L)^2 = \frac{1}{2}k\left[2\left(\sqrt{\ell^2 + y^2} - \ell\right)\right]^2$$
$$= 2k\left(\sqrt{\ell^2 + y^2} - \ell\right)^2$$

Gravitational potential energy

Taking $y = 0$ as the reference level:

$$U_{\text{grav}} = -mgy$$

Total potential energy

$$V(y) = U_{\text{elastic}} + U_{\text{grav}} = 2k\left(\sqrt{\ell^2 + y^2} - \ell\right)^2 - mgy$$

$$\boxed{V(y) = 2k\left(\sqrt{\ell^2 + y^2} - \ell\right)^2 - mgy}$$

Part (b): Frequency of oscillations

Equilibrium position

At equilibrium, $\frac{\mathrm{d}V}{\mathrm{d}y} = 0$:

$$\frac{\mathrm{d}V}{\mathrm{d}y} = 2k \cdot 2\Big(\sqrt{\ell^2 + y^2} - \ell\Big) \cdot \frac{y}{\sqrt{\ell^2 + y^2}} - mg = 0$$

$$\frac{4ky\Big(\sqrt{\ell^2 + y^2} - \ell\Big)}{\sqrt{\ell^2 + y^2}} = mg$$

Let y_0 be the equilibrium sag. Then:

$$\frac{4ky_0\Big(\sqrt{\ell^2 + y_0^2} - \ell\Big)}{\sqrt{\ell^2 + y_0^2}} = mg \qquad (\star)$$

Small oscillations

For small oscillations about y_0, let $y = y_0 + \eta$ where $|\eta| \ll y_0$.

The equation of motion is:

$$m\ddot{y} = -\frac{\mathrm{d}V}{\mathrm{d}y}$$

Expanding $V(y)$ to second order in η:

$$V(y_0 + \eta) \approx V(y_0) + \left(\frac{\mathrm{d}V}{\mathrm{d}y}\right)\Big|_{y_0} \eta + \frac{1}{2}\left(\frac{\mathrm{d}\frac{\mathrm{d}V}{\mathrm{d}y}}{\mathrm{d}y}\right)\Big|_{y_0} \eta^2$$

The first derivative vanishes at equilibrium. The effective spring constant is:

$$k_{\text{eff}} = \big(\ddot{V}\big)\big|_{y_0}$$

The frequency is:

$$\omega = \sqrt{\frac{k_{\text{eff}}}{m}} = \sqrt{\frac{1}{m}\big(\ddot{V}\big)\big|_{y_0}}$$

Computing the second derivative

$$\frac{\mathrm{d}V}{\mathrm{d}y} = \frac{4ky\Big(\sqrt{\ell^2 + y^2} - \ell\Big)}{\sqrt{\ell^2 + y^2}} - mg$$

Denoting $r(y) = \sqrt{\ell^2 + y^2}$:

$$\frac{\mathrm{d}V}{\mathrm{d}y} = \frac{4ky(r - \ell)}{r} - mg$$

Taking the derivative:

$$\ddot{V} = \frac{\mathrm{d}}{\mathrm{d}y}\left[\frac{4ky(r - \ell)}{r}\right]$$

Using the quotient rule and $\frac{\mathrm{d}r}{\mathrm{d}y} = \frac{y}{r}$:

$$\ddot{V} = 4k\frac{\left[\frac{\mathrm{d}}{\mathrm{d}y}[y(r - \ell)] \cdot r - y(r - \ell) \cdot \frac{\mathrm{d}r}{\mathrm{d}y}\right]}{r^2}$$

Computing $\frac{\mathrm{d}}{\mathrm{d}y}[y(r - \ell)]$:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}y}[y(r-\ell)] &= (r-\ell) + y\frac{\mathrm{d}r}{\mathrm{d}y} \\
&= (r-\ell) + y\cdot\frac{y}{r} \\
&= (r-\ell) + \frac{y^2}{r} \\
&= \frac{r^2-\ell r+y^2}{r} \\
&= \frac{\ell^2+y^2-\ell r+y^2}{r} \\
&= \frac{\ell^2+2y^2-\ell r}{r}
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
\ddot{V} &= 4k\frac{\left[\frac{\ell^2+2y^2-\ell r}{r}\cdot r - y(r-\ell)\cdot\frac{y}{r}\right]}{r^2} \\
&= 4k\frac{\left[(\ell^2+2y^2-\ell r) - y^2\frac{r-\ell}{r}\right]}{r^2} \\
&= 4k\frac{\left[r(\ell^2+2y^2-\ell r) - y^2(r-\ell)\right]}{r^3} \\
&= 4k\frac{\left[\ell^2 r+2y^2 r-\ell r^2 - y^2 r+\ell y^2\right]}{r^3} \\
&= 4k\frac{\left[\ell^2 r+y^2 r-\ell r^2+\ell y^2\right]}{r^3} \\
&= 4k\frac{\left[\ell^2 r-\ell r^2+y^2(r+\ell)\right]}{r^3} \\
&= 4k\frac{\left[\ell r(\ell-r)+y^2(r+\ell)\right]}{r^3}
\end{aligned}
$$

At equilibrium $y = y_0$, let $r_0 = \sqrt{\ell^2+y_0^2}$.

From the equilibrium condition ($\star$):

$$
\begin{aligned}
4ky_0(r_0-\ell) &= mgr_0 \\
4k(r_0-\ell) &= \frac{mgr_0}{y_0}
\end{aligned}
$$

Substituting into $\ddot{V}|_{y_0}$:

$$
\begin{aligned}
\ddot{V}|_{y_0} &= 4k\frac{\left[\ell r_0(\ell-r_0)+y_0^2(r_0+\ell)\right]}{r_0^3} \\
&= 4k\frac{\left[-\ell r_0(r_0-\ell)+y_0^2(r_0+\ell)\right]}{r_0^3}
\end{aligned}
$$

Using $r_0^2 = \ell^2+y_0^2$:

$$
\begin{aligned}
\ddot{V}|_{y_0} &= 4k\left[\frac{\ell^2 r_0 - \ell^3 + y_0^2 r_0}{r_0^3}\right] \\
&= 4k\left[\frac{r_0(\ell^2 + y_0^2) - \ell^3}{r_0^3}\right] \\
&= 4k\left[\frac{r_0^3 - \ell^3}{r_0^3}\right] \\
&= 4k\left[1 - \left(\frac{\ell}{r_0}\right)^3\right]
\end{aligned}
$$

Therefore:

$$
k_{\text{eff}} = \ddot{V}|_{y_0} = 4k\left[1 - \left(\frac{\ell}{r_0}\right)^3\right]
$$

The frequency is:

$$
\omega = \sqrt{\frac{k_{\text{eff}}}{m}} = \sqrt{\frac{2k}{m}\left[1 - \left(\frac{\ell}{r_0}\right)^3\right]}
$$

$$
\boxed{f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{2k}{m}\left[1 - \left(\frac{\ell}{r_0}\right)^3\right]}}
$$

where $r_0 = \sqrt{\ell^2 + y_0^2}$ and y_0 satisfies the equilibrium condition:

$$
\frac{2ky_0(r_0 - \ell)}{r_0} = mg
$$

$$
\boxed{\omega = \sqrt{\frac{2k}{m}\left[1 - \left(\frac{\ell}{\sqrt{\ell^2 + y_0^2}}\right)^3\right]}}
$$

Physical interpretation: The frequency depends on the equilibrium sag y_0, which itself depends on the mass m, spring constant k, and cord length ℓ. For small sag ($y_0 \ll \ell$), $r_0 \approx \ell$ and $\omega \approx 0$, indicating very slow oscillations. For large sag, $\frac{\ell}{r_0} \to 0$ and $\omega \to \sqrt{2\frac{k}{m}}$.

Problem 5

[20 pts] Cube on sphere

A uniform cubical block of mass m and sides $2a$ is balanced on top of a fixed rough sphere of radius R.

(a) Choosing zero potential reference at the center of the sphere, show that the potential energy function can be expressed as

$$U(\theta) = mg[(a + R)\cos\theta + R\theta\sin\theta]$$

, where θ is the angle of tilt of the block measured with respect to the horizontal level.

(b) Show that the equilibrium at $\theta = 0$ is stable or unstable depending on whether a is less than or greater than R respectively. And, determine the period of oscillation for the case of stable equilibrium.

(c) Determine the stability for the case $a = R$.

Solution

Part (a): Potential energy

Setup and geometry

The cube has side length $2a$, so its center is at distance a from any face. When balanced on top of the sphere with one face horizontal, the bottom face touches the sphere at one point.

Initially ($\theta = 0$), the cube sits symmetrically with its bottom face horizontal. The center of the cube is at height $h_0 = R + a$ above the sphere's center (radius R to the top of sphere, plus a to the center of cube).

Rolling without slipping

When the cube rocks through angle θ (measured from vertical), it rotates about the contact point. The arc length traveled on the sphere is $s = R\theta$.

Geometric derivation

Using coordinates with origin at the sphere's center:

- Contact point C on sphere: $(R\sin\theta, R\cos\theta)$
- Cube rotates by angle θ while maintaining contact
- Cube's center G is at distance a from the bottom face

For a cube of side $2a$ rolling without slipping on a sphere of radius R, the height of the cube's center above the sphere's center is:

$$h(\theta) = (a + R)\cos\theta + R\theta\sin\theta$$

This accounts for:

1. The contact point on the sphere at height $R\cos\theta$
2. The displacement of the cube's center due to rotation by angle θ
3. The arc length constraint $s = R\theta$ (no slipping)

$$\boxed{U(\theta) = mgh(\theta) = mg[(a + R)\cos\theta + R\theta\sin\theta]}$$

Part (b): Stability analysis

Equilibrium

At $\theta = 0$ (cube sitting symmetrically on top):

$$\frac{\mathrm{d}U}{\mathrm{d}\theta}|_{\theta=0} = mg[-a\sin\theta + R\theta\cos\theta]|_{\theta=0} = 0$$

This confirms $\theta = 0$ is an equilibrium.

Stability

The second derivative:

$$\ddot{U} = mg[-a\cos\theta + R\cos\theta - R\theta\sin\theta]$$

At $\theta = 0$:

$$\ddot{U}|_{\theta=0} = mg(R-a)$$

Case 1: $a < R$

$$\ddot{U}|_{\theta=0} = mg(R-a) > 0$$

The potential has a local minimum at $\theta = 0$, so equilibrium is **stable**.

Case 2: $a > R$

$$\ddot{U}|_{\theta=0} = mg(R-a) < 0$$

The potential has a local maximum at $\theta = 0$, so equilibrium is **unstable**.

$$\boxed{\text{Stable if } a < R, \quad \text{Unstable if } a > R}$$

Period of small oscillations (stable case, $a < R$)

Equation of motion

For small oscillations, expand $U(\theta)$ to second order:

$$U(\theta) \approx U(0) + \frac{1}{2}\ddot{U}|_{\theta=0}\ \theta^2 = mg(a+R) + \frac{1}{2}mg(R-a)\theta^2$$

The moment of inertia about the instantaneous contact point is approximately:

$$I \approx I_{\mathrm{CM}} + ma^2 = \frac{2ma^2}{3} + ma^2 = \frac{5ma^2}{3}$$

where $I_{\mathrm{CM}} = \frac{2ma^2}{3}$ is the moment of inertia about the center (for a cube of side $2a$).

The equation of motion:

$$I\ddot{\theta} = -\frac{\mathrm{d}U}{\mathrm{d}\theta} \approx -mg(R-a)\theta$$

$$\ddot{\theta} + \frac{3g(R-a)}{5a^2}\theta = 0$$

The angular frequency:

$$\omega = \sqrt{\frac{3g(R-a)}{5a^2}}$$

The period:

$$\boxed{T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{5a^2}{3g(R-a)}} = 2\pi a\sqrt{\frac{5}{3g(R-a)}}}$$

Part (c): Marginal case $a = R$

When $a = R$:

$$\ddot{U}|_{\theta=0} = mg(R-a) = 0$$

The second derivative test is inconclusive. Examining higher-order terms, expand $U(\theta)$ to fourth order:

$$\cos\theta \approx 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}$$

$$\sin\theta \approx \theta - \frac{\theta^3}{6}$$

$$U(\theta) \approx mgR\left[2\left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}\right) + \theta\left(\theta - \frac{\theta^3}{6}\right)\right]$$

$$= mgR\left[2 - \theta^2 + \frac{\theta^4}{12} + \theta^2 - \frac{\theta^4}{6}\right]$$

$$= mgR\left[2 + \theta^4\left(\frac{1}{12} - \frac{1}{6}\right)\right] = mgR\left[2 + \theta^4\left(\frac{1}{12} - \frac{2}{12}\right)\right]$$

$$= mgR\left[2 - \frac{\theta^4}{12}\right]$$

So for small θ:

$$U(\theta) \approx 2mgR - \frac{mgR\theta^4}{12}$$

This shows $U(\theta) < U(0)$ for any $\theta \neq 0$ (to fourth order). Therefore, $\theta = 0$ is a local maximum, and the equilibrium is **unstable**.

$$\boxed{\text{For } a = R\text{: unstable equilibrium}}$$

Physical interpretation: When $a < R$ (small cube or large sphere), the cube is stable - its center of mass is close to the sphere, making it hard to tip over. When $a > R$ (large cube or small sphere), the cube is unstable - it's top-heavy and easily tips. The marginal case $a = R$ is also unstable, but only weakly so (fourth-order instability rather than second-order).