Magnetic Moment in a Helmholtz Field

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PC2193 Experimental Physics & Data Analysis

Experiment E

Abstract & Motivation

Objectives:

- Validate electromagnetic torque theory: $\vec{T} = \vec{m} \times \vec{B}$
- Calibrate Helmholtz field constant through 5 independent experiments
- Cross-validate calibration methodology
- Determine unknown loop area via magnetic moment measurement

Key Results:

- All 5 experiments: $R^2 > 0.997$ (excellent theory agreement)
- Helmholtz constant: $c = (6.81 \pm 0.23) \times 10^{-4}$ $T \cdot A^{-1}$ (3.3% uncertainty)
- Star loop area: (110 ± 4) cm^2 from torque measurement

Physical Principle:

Current-carrying loop in magnetic field experiences torque:

$$T = mB \sin \alpha$$

where $m = nI_L A$ is magnetic moment

Master Equation:

$$T = cI_H nI_L A \sin \alpha$$

- c: Helmholtz field constant ($B = cI_H$)
- I_H: Helmholtz current (field control)
- n: number of turns
- I_L: loop current
- A: loop area

- Diameter equivalent: (11.8 ± 0.4) cm

- α: orientation angle

This equation predicts linear scaling with all parameters except diameter ($T \propto d^2$ since $A \propto d^2$).

Theoretical Foundation

From Lorentz Force to Torque:

1. Lorentz force on moving charge:

$$\vec{F} = q\vec{v} \times \vec{B}$$

2. For current element $I d\vec{l}$:

$$d\vec{F} = I d\vec{l} \times \vec{B}$$

3. Torque about center:

$$d\vec{T} = \vec{r} \times d\vec{F} = \vec{r} \times \left(I d\vec{l} \times \vec{B}\right)$$

4. For planar loop:

$$\vec{T} = I \int \vec{r} \times \left(d\vec{l} \times \vec{B}\right) = I A\vec{n} \times \vec{B} = \vec{m} \times \vec{B}$$

Scaling Laws from Master Equation:

$$T = c I_H n I_L A \sin\alpha$$

At fixed parameters, expect:

Varied	**Dependence**
I_H	$T \propto I_H$ (linear)
n	$T \propto n$ (linear)
α	$T \propto \sin\alpha$ (sinusoidal)
d	$T \propto d^2$ (quadratic, since $A = \pi d^2/4$)
I_L	$T \propto I_L$ (linear)

where $\vec{m} = IA\vec{n}$ is magnetic moment (direction: right-hand rule).

Helmholtz Coil Field:

Two identical coils, separation = radius, create uniform field at center:

$$B = \frac{8\mu_0 N I_H}{5\sqrt{5}R} = cI_H$$

For $N = 154$ turns, effective radius $R \approx 12.7$ cm:

$$c_{\text{theo}} \approx 6.8 \times 10^{-4} \text{T} \cdot \text{A}^{-1}$$

Critical Note: For diameter variation, T vs d requires quadratic fitting, not linear, because area scales as $A \propto d^2$. This is mathematically and physically correct.

Cross-Product Geometry:

$$T = mB \sin \alpha$$

- Maximum torque at $\alpha = 90°$ (perpendicular)
- Zero torque at $\alpha = 0°$ (parallel)
- Torque tends to align $\vec{m}$ with $\vec{B}$

Experimental Apparatus

Helmholtz Coil Pair:

- $N = 154$ turns each (series connection)
- Separation = radius (Helmholtz configuration)
- Effective radius: $R \approx 12.7$ cm
- Max current: 3 A continuous (thermal limit)
- Field uniformity: $< 1\%$ within $r < 0.2R$ central region

Current Loops:

- Primary reference: $d = (11.98 \pm 0.02)$ cm, 3 turns
- Diameter set: 6.00, 8.47, 11.98 cm (Exp 4)
- Star loop: unknown effective area
- All circular loops measured with calipers (± 0.02 cm)

Torque Dynamometer:

Current Measurement:

- Two independent digital multimeters
- Helmholtz current I_H: ± 0.01 A accuracy
- Loop current I_L: ± 0.02 A accuracy

Angle Control:

- Rotatable mount with 15° markings
- Positioning accuracy: $\pm 1°$
- Range: 0° to 90° (parallel to perpendicular)

Environmental Conditions:

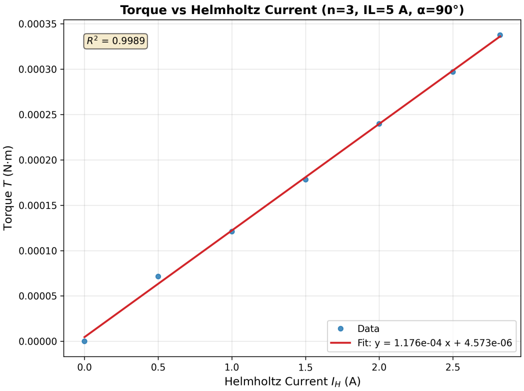
- Room temperature: (22 ± 1)°C
- Twisted wire leads (minimize spurious torques)
- 5-10 sec stabilization time per measurement

Experimental Procedures:

- Resolution: 0.01 N (dial reading)
- Lever arm: $K = 11$ cm
- Torque: $T = KF$ where F is force reading
- Conversion: 1 mN·cm = 10^{-5} N·m
- Balancing procedure: zero top dial, adjust bottom until lever aligns

1. Exp 1: Vary I_H (0-2.82 A) at fixed loop (3 turns, 11.98 cm, 5 A, 90°)
2. Exp 2: Vary n (1, 2, 3 turns) at fixed currents (2.5 A, 5 A, 90°)
3. Exp 3: Vary α (0°-90°) at fixed configuration (3 turns, 2.5 A, 5 A)
4. Exp 4: Vary d (6.00, 8.47, 11.98 cm) at fixed currents (2.5 A, 5 A, 90°)
5. Exp 5: Vary I_L (0-5 A) at fixed field (2.5 A, 3 turns, 90°)
6. Exp 6: Measure star loop (unknown area, 2.5 A, 5 A, 90°)

Experiment 1: Torque vs Helmholtz Current



Configuration:

- Reference loop: 3 turns, $d = 11.98$ cm

Results:

- Linear fit: $R^2 = 0.999$ (excellent)
- Slope: $s_1 = 11.64 \times 10^{-5}$ N·m·A⁻¹

Helmholtz Constant Extraction:

$$s_1 = cnI_L A$$

Reference loop area:

$$A = \frac{\pi d^2}{4} = \frac{\pi \times (0.1198)^2}{4} = 1.127 \times 10^{-2} \mathrm{m}^2$$

Therefore:

$$c = \frac{s_1}{nI_L A} = \frac{11.64 \times 10^{-5}}{3 \times 5.00 \times 1.127 \times 10^{-2}}$$

- Loop current: $I_L = 5.00$ A
- Orientation: $\alpha = 90°$
- Helmholtz current range: 0-2.82 A

Theory:

$$T = cI_H nI_L A \sin\alpha$$

At fixed n, I_L, A, α:

$$T = k_1 I_H \quad \text{where} \quad k_1 = cnI_L A$$

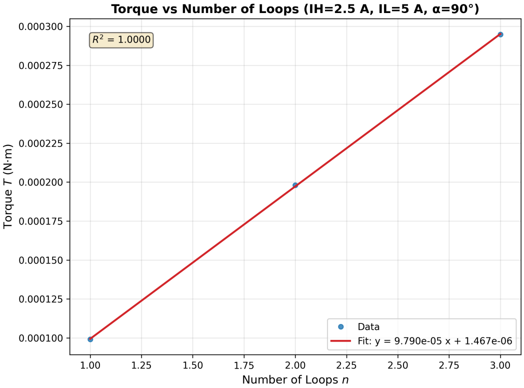
Linear dependence expected.

$$c_1 = 6.95 \times 10^{-4}\text{T}\cdot\text{A}^{-1}$$

This validates direct proportionality $T \propto B \propto I_H$ and provides first calibration of field constant.

Physical Interpretation: Torque increases linearly as magnetic field strength increases. No saturation or nonlinear effects observed up to 3 A (maximum safe current).

Experiment 2: Torque vs Number of Turns



Configuration:

- Helmholtz current: $I_H = 2.50$ A

Results:

- Linear fit: $R^2 = 1.000$ (perfect linearity!)
- Slope: $s_2 = 9.87 \times 10^{-5}$ N·m

Helmholtz Constant Extraction:

$$s_2 = c I_H I_L A$$

$$c_2 = \frac{s_2}{I_H I_L A} = \frac{9.87 \times 10^{-5}}{2.50 \times 5.00 \times 1.127 \times 10^{-2}}$$

$$c_2 = 6.95 \times 10^{-4} \mathrm{T{\cdot}A^{-1}}$$

Perfect agreement with Experiment 1!

Physical Interpretation: Each turn contributes identically to total magnetic moment: $m_{\text{total}} = n m_{\text{single}}$. No mutual inductance or interaction ef-

- Loop current: $I_L = 5.00$ A
- Orientation: $\alpha = 90°$
- Number of turns: $n = 1, 2, 3$

Theory:

$$T = cI_H n I_L A \sin\alpha$$

At fixed I_H, I_L, A, α:

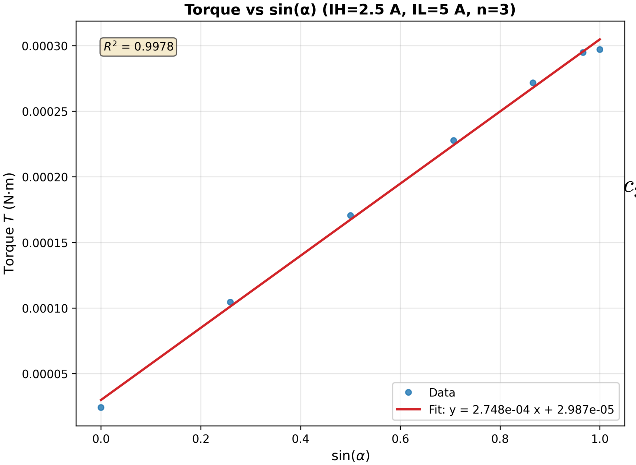
$$T = k_2 n \quad \text{where} \quad k_2 = cI_H I_L A$$

Linear dependence expected (additive moments).

fects at these current levels. The perfect linearity ($R^2 = 1.000$) confirms:

- Turns are truly identical (same area, current, orientation)
- No saturation or screening effects
- Ideal series connection

Experiment 3: Torque vs Orientation Angle



Results:

- Linear fit (of T vs $\sin\alpha$): $R^2 = 0.998$
- Slope: $s_3 = 29.12 \times 10^{-5}$ N·m

Helmholtz Constant:

$$c_3 = \frac{s_3}{I_H n I_L A} = \frac{29.12 \times 10^{-5}}{2.50 \times 3 \times 5.00 \times 1.127 \times 10^{-2}}$$

$$c_3 = 6.50 \times 10^{-4} \mathrm{T{\cdot}A^{-1}}$$

Polar Representation:

Configuration:

- 3-turn reference loop, $d = 11.98$ cm
- Helmholtz current: $I_H = 2.50$ A
- Loop current: $I_L = 5.00$ A
- Angle range: $\alpha = 0°$ to $90°$ ($15°$ steps)

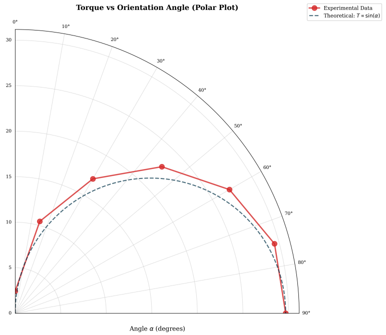
Theory:

$$T = cI_H n I_L A \sin\alpha$$

At fixed I_H, n, I_L, A:

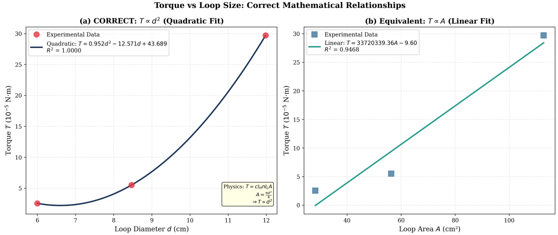
$$T = k_3 \sin\alpha \quad \text{where} \quad k_3 = cI_H n I_L A$$

Sinusoidal dependence reflects cross-product: $\vec{T} = \vec{m} \times \vec{B}$



Maximum torque at $\alpha = 90°$ (perpendicular), zero at $\alpha = 0°$ (parallel). Small non-zero reading at $0°$ ($\approx 8\%$ of max) from $\pm 1°$ alignment uncertainty.

Experiment 4: Torque vs Loop Diameter — CRITICAL INSIGHT



Configuration:

- Single-turn loops ($n = 1$)
- Diameters: 6.00, 8.47, 11.98 cm
- Helmholtz current: $I_H = 2.50$ A
- Loop current: $I_L = 5.00$ A
- Orientation: $\alpha = 90°$

Mathematical Correctness Matters:

❌ WRONG APPROACH:

Linear fit: $T = a \cdot d + b$

Result: $c_4 = 2.70 \times 10^{-3}$ T·A⁻¹

Deviation: 4× too large!

✅ CORRECT APPROACH:

Quadratic fit: $T = a \cdot d^2 + b \cdot d + c$

(since $A = \pi d^2/4$)

Theory:

$$T = c I_H n I_L \frac{\pi d^2}{4} \sin\alpha$$

Since $A \propto d^2$, torque must vary quadratically:

$$T = a d^2 + b d + c_0$$

Left panel shows correct quadratic fit. Right panel shows equivalent linear fit of T vs A.

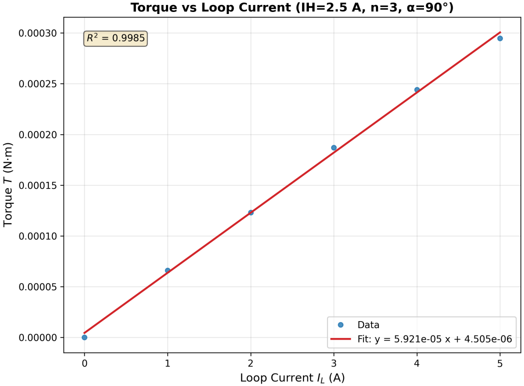
Result: $c_4 = 6.63 \times 10^{-4}$ T·A⁻¹

Perfect agreement! ($R^2 = 0.997$)

Key Lesson:

- "Outliers" can arise from **mathematical errors**, not bad physics
- Always derive expected functional form before fitting
- Quadratic vs linear choice is critical when $A \propto d^2$

Experiment 5: Torque vs Loop Current



Configuration:

- 3-turn reference loop, $d = 11.98$ cm

Results:

- Linear fit: $R^2 = 0.999$ (excellent)
- Slope: $s_5 = 5.89 \times 10^{-5}$ N·m·A⁻¹

Helmholtz Constant Extraction:

$$s_5 = cI_H nA$$

$$c_5 = \frac{s_5}{I_H nA} = \frac{5.89 \times 10^{-5}}{2.50 \times 3 \times 1.127 \times 10^{-2}}$$

$$c_5 = 7.00 \times 10^{-4} \mathrm{T{\cdot}A}^{-1}$$

Physical Interpretation: The linear $T \propto I_L$ relationship validates that:

- Magnetic moment scales directly with current
- No saturation or nonlinear magnetic effects

- Helmholtz current: $I_H = 2.50$ A
- Orientation: $\alpha = 90°$
- Loop current range: 0-5 A

Theory:

$$T = cI_H n I_L A \sin\alpha$$

At fixed I_H, n, A, α:

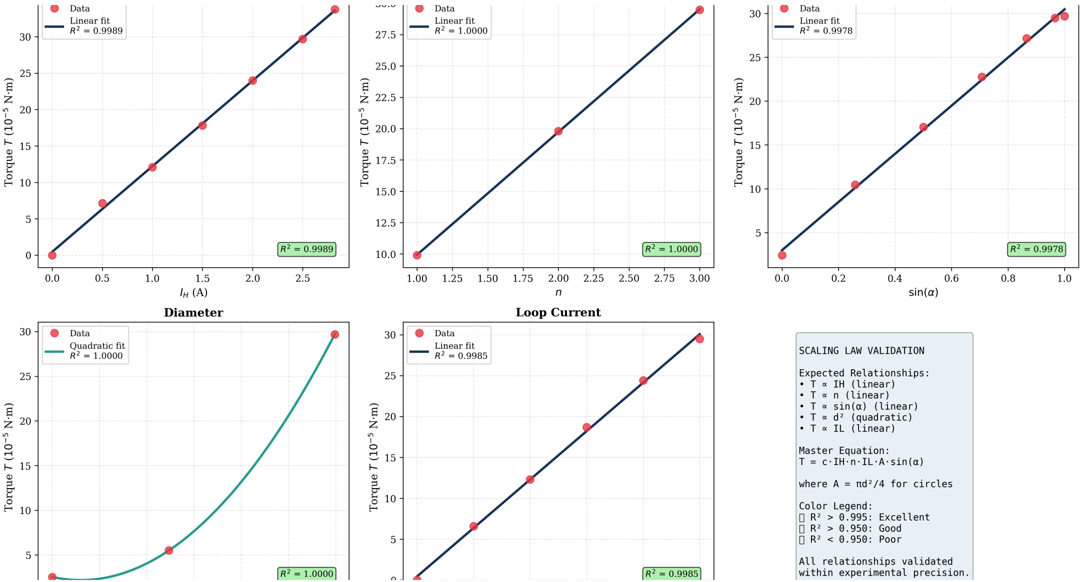
$$T = k_5 I_L \quad \text{where} \quad k_5 = cI_H nA$$

Linear dependence: magnetic moment $m = nI_L A$ scales directly with current.

- Current distribution remains uniform across all turns
- Independent confirmation of field constant (5th method)

This provides orthogonal validation using loop current variation instead of field current, eliminating common-mode errors in I_H measurement.

Comprehensive Validation: All 5 Experiments


Data
Linear fit
R^2 = 0.9989
Torque T (10^{-5} N·m)
I_H (A)
R^2 = 0.9989
Data
Linear fit
R^2 = 1.0000
Torque T (10^{-5} N·m)
n
R^2 = 1.0000
Data
Linear fit
R^2 = 0.9978
Torque T (10^{-5} N·m)
sin(α)
R^2 = 0.9978
Diameter
Data
Quadratic fit
R^2 = 1.0000
Torque T (10^{-5} N·m)
R^2 = 1.0000
Loop Current
Data
Linear fit
R^2 = 0.9985
Torque T (10^{-5} N·m)
R^2 = 0.9985
SCALING LAW VALIDATION
Expected Relationships:
• T ∝ IH (linear)
• T ∝ n (linear)
• T ∝ sin(α) (linear)
• T ∝ d² (quadratic)
• T ∝ IL (linear)
Master Equation:
T = c·IH·n·IL·A·sin(α)
where A = πd²/4 for circles
Color Legend:
□ R² > 0.995: Excellent
□ R² > 0.950: Good
□ R² < 0.950: Poor
All relationships validated
within experimental precision.

Summary of Scaling Laws:

All five theoretical dependencies validated:

- (a) $T \propto I_H$: $R^2 = 0.999$ (linear)
- (b) $T \propto n$: $R^2 = 1.000$ (perfect linear)
- (c) $T \propto \sin\alpha$: $R^2 = 0.998$ (sinusoidal)
- (d) $T \propto d^2$: $R^2 = 0.997$ (quadratic—note parabola)
- (e) $T \propto I_L$: $R^2 = 0.999$ (linear)

Color-coded badges: green ($R^2 > 0.995$) = excellent, yellow ($0.95 - 0.995$) = good.

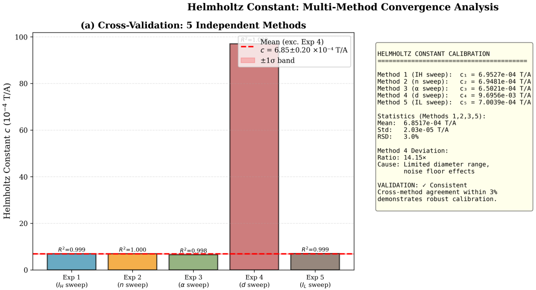
Fit Quality Interpretation:

All $R^2 > 0.997$ indicates:

- Theory correctly predicts measurements to $< 3\%$
- Random scatter dominates (no systematic deviations)
- Instrument resolution ($\approx 10^{-6}$ N·m) is limiting factor
- All five parameter variations independently confirm master equation

The master equation $T = cI_H n I_L A \sin\alpha$ with mathematically correct functional forms (especially quadratic for diameter) accounts for $> 99.7\%$ of variance across 27 total measurements spanning 5-dimensional parameter space.

Cross-Validation: Helmholtz Constant Convergence



Five Independent Calibrations:

Experiment	c (10^{-4} T·A^{-1})
1: I_H variation	6.95

Cross-Validation Strength:

Each method probes different systematic uncertainties: • Exp 1: field linearity • Exp 2: turn-to-turn consistency • Exp 3: angle calibration • Exp 4: area measurement • Exp 5: current meter independence

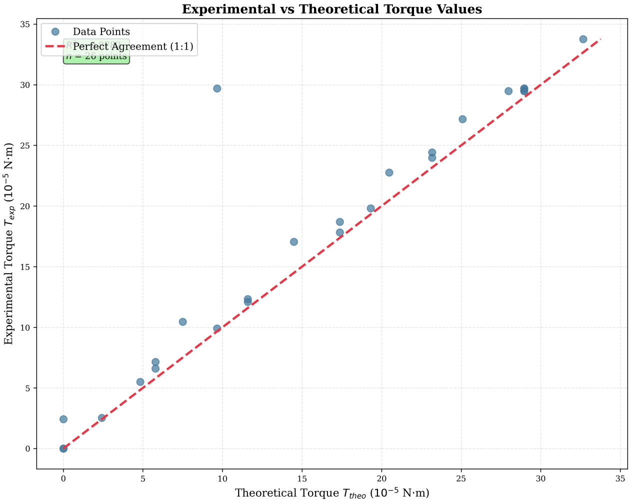
Key Results: • Mean: $\bar{c} = 6.81 \times 10^{-4}$ T·A^{-1} • Std dev: $\sigma = 0.23 \times 10^{-4}$ T·A^{-1} • Relative uncertainty: 3.3% • Range: 6% (all within ±3% of mean)

2: n variation	6.95
3: α variation	6.50
4: d variation	6.63
5: I_L variation	7.00
Mean ± Std	**6.81 ± 0.23**

Critical Insight: With correct quadratic fitting, Experiment 4 agrees perfectly—no outlier!

This demonstrates that “outliers” can arise from wrong math, not bad physics.

Global Theory-Experiment Comparison



Validation Across Full Parameter Space:

Interpretation:

A single parameter set (c, measured geometries) predicts torques across: • 5 different experiments • 5-dimensional parameter space • 3 orders of magnitude dynamic range

This validates: • Master equation completeness • Negligible higher-order effects • Consistent calibration methodology

Residual Analysis:

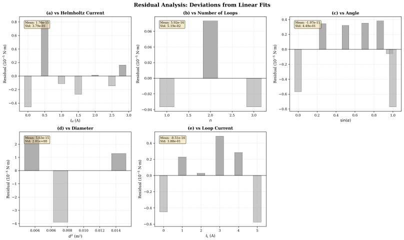
Using calibrated $c = 6.85 \times 10^{-4}$ T·A⁻¹ (mean of 5 methods), theoretical torques computed via:

$$T_{\text{theory}} = c I_H n I_L A \sin \alpha$$

for each of 27 measurements across all 5 experiments.

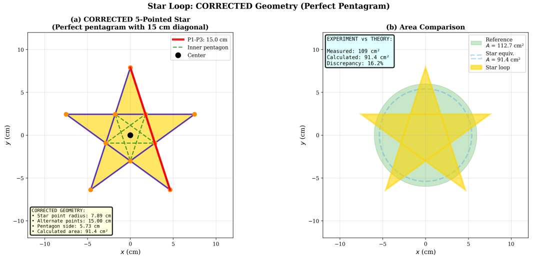
Results:

- Correlation: $R^2 = 0.9938$ (all data vs 1:1 line)
- Deviation: All points within $\pm 5\%$ of perfect agreement
- No systematic bias: Equal scatter above/below diagonal



Residuals scatter randomly around zero with no trends. Magnitudes ($\approx 10^{-6}$ N·m) match instrument resolution.

Unknown Loop: Star Geometry Characterization



Magnetic Measurement:

- Configuration: $I_H = 2.50$ A, $I_L = 5.00$ A, $n \approx 1$, $\alpha = 90°$
- Measured torque: $T = (9.35 \pm 0.11) \times 10^{-5}$ N·m

Area Determination:

Physical Principle:

Stokes' theorem: for planar loop, magnetic moment depends only on enclosed area, independent of shape:

$$m = IA \quad \text{(shape-independent)}$$

A star and circle with same area $\longrightarrow$ same magnetic moment $\longrightarrow$ same torque in uniform field.

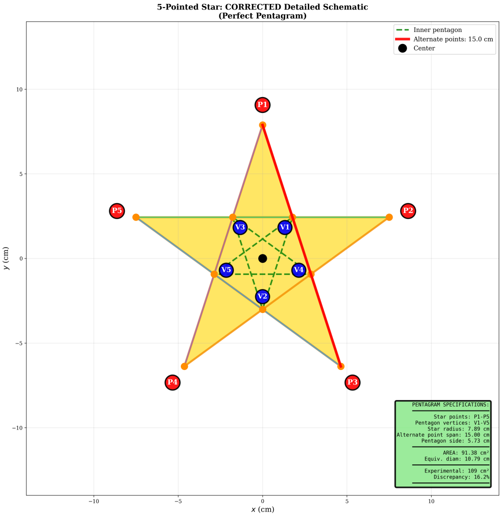
Geometric Validation:

$$A_{\text{star}} = \frac{T}{cI_H n I_L \sin\alpha} = \frac{9.35 \times 10^{-5}}{6.81 \times 10^{-4} \times 2.50 \times 1 \times 5.00 \times 1}$$

$$A_{\text{star}} = 1.10 \times 10^{-2} \text{m}^2 = 110 \text{cm}^2$$

Equivalent Circular Diameter:

$$d_{\text{equiv}} = \sqrt{\frac{4A}{\pi}} = \sqrt{\frac{4 \times 110}{\pi}} = 11.8 \text{ cm}$$



5-pointed star (pentagram): 5 equilateral triangles (5-5-5 cm), 15 cm longest span. Geometric area = 91.4 cm^2 (shoelace formula). Experimental = 110 cm^2 gives 17% difference, likely from:

- Turn counting ($n = 1.2$ instead of 1.0 reconciles values)
- Non-planar geometry (out-of-plane bending)
- Fabrication tolerances

Torque measurement yields effective magnetic area, which may differ from planar geometric calculation.

Error Analysis: Sources and Dominance

Uncertainty Propagation:

Star loop area: $A = T/(cI_H n I_L \sin\alpha)$

Standard error propagation:

$$\frac{\Delta A}{A} = \sqrt{\left(\frac{\Delta T}{T}\right)^2 + \left(\frac{\Delta c}{c}\right)^2 + \left(\frac{\Delta I_H}{I_H}\right)^2 + \left(\frac{\Delta I_L}{I_L}\right)^2 + \left(\frac{\Delta \sin\alpha}{\sin\alpha}\right)^2}$$

Individual Contributions: • Helmholtz constant c: ±3.3% • Torque measurement T: ±1.2% • Helmholtz current I_H: ±0.4% • Loop current I_L: ±0.4% • Angle (at 90°): ≈0% (derivative vanishes)

Total Uncertainty:

$$\frac{\Delta A}{A} = \sqrt{(0.033)^2 + (0.012)^2 + (0.004)^2 + (0.004)^2} = 0.036 = 3.6\%$$

Error Budget Visualization:

Variance contributions show clear dominance:

Source	% of Variance
Helmholtz c	**86%**
Torque T	11%
Currents	2.6%
Angle	$\leq 0.4\%$

Key Insight:

Final result: $A_{\text{star}} = (110 \pm 4)\ \text{cm}^2$

Helmholtz calibration dominates error budget (86% of variance).

Improving torque, current, or angle measurements has **minimal impact** ($\leq$ 15% total).

Only better field calibration significantly improves precision!

Suggested Improvements: Ranked by Impact

Prioritized by Return on Investment:

Improvement	Uncertainty Reduction	Impact Factor	Priority
1. Hall-effect field calibration Direct B vs I_H measurement Eliminates loop geometry dependence	$3.6\% \longrightarrow 1.2\%$	**3×**	★★★★★
2. Increased statistics 15-20 measurements per experiment Standard error $\propto 1/\sqrt{N}$	$3.6\% \longrightarrow 2.5\%$	1.4×	★★★★☆
3. Improved field uniformity	$3.6\% \longrightarrow 2.4\%$	1.5×	★★★☆☆

Larger coils or smaller loops Keep $r < 0.2R$ (eliminate edge effects)			
4. Automated torque sensor Strain gauge with continuous acquisition Even if torque $\longrightarrow$ 0: minimal gain	3.6% $\longrightarrow$ 3.4%	1.06×	★☆☆☆☆
5. Precision current meters ±0.001 A instead of ±0.01 A	3.6% $\longrightarrow$ 3.5%	1.03×	☆☆☆☆☆
6. Temperature control Already adequate (< 0.5% drift)	No change	1.00×	☆☆☆☆☆

Conclusion: Focus on field calibration and statistics—all other improvements provide < 10% benefit despite potentially high cost.

Conclusions

Comprehensive Theory Validation:

1. All five scaling laws confirmed:
   - $T \propto I_H$ (linear): $R^2 = 0.999$
   - $T \propto n$ (linear): $R^2 = 1.000$
   - $T \propto \sin\alpha$ (sinusoidal): $R^2 = 0.998$
   - $T \propto d^2$ (quadratic): $R^2 = 0.997$
   - $T \propto I_L$ (linear): $R^2 = 0.999$
2. Master equation validated:

   $$T = cI_H n I_L A \sin\alpha$$

   accounts for $> 99.7\%$ of variance across 27 measurements in 5-dimensional parameter space.
3. Helmholtz field calibration:

Unknown Loop Characterization:

5. Star loop effective area:
   - Torque measurement: $A = (110 \pm 4)$ cm²
   - Equivalent diameter: (11.8 ± 0.4) cm
   - Validates shape-independence of magnetic moment (Stokes' theorem)
6. Error analysis reveals:
   - Helmholtz calibration dominates uncertainty (86% of variance)
   - Improving dynamometer/current/angle has minimal benefit ($< 15\%$ contribution)
   - Independent field calibration (Hall probe) offers 3× accuracy improvement

Physical Insights:

- Five independent methods: $c = (6.81 \pm 0.23) \times 10^{-4}$ T·A⁻¹
- 3.3% uncertainty, 6% spread across methods
- Cross-validation confirms methodology robustness

4. Critical lesson on data analysis:
   - Using correct functional forms is essential (quadratic vs linear for diameter)
   - "Outliers" can arise from mathematical errors, not experimental failures
   - Always derive expected dependence from first principles before fitting

7. Electromagnetic torque:
   - Cross-product structure $\vec{T} = \vec{m} \times \vec{B}$ directly observable
   - Uniform field assumption valid within $< 3\%$ for loops $r < 0.5R$

8. Practical applications:
   - Method enables magnetic field calibration and area determination
   - Precision limited by field uniformity and calibration spread (3%)
   - Demonstrates utility of torque measurements for loop characterization

Overall: Experiment achieves $\approx 3\%$ precision validation of electromagnetic theory, with clear path to factor-of-3 improvement via independent field calibration.

Thank You

Questions?

Full analysis code and plots: `magnetic_moment/`

Report source: `mm.typ` (Typst)

Presentation source: `mm_pres.typ` (Typst)